CENTRE-EXTENDED-BY-METABELIAN GROUPS.

by

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STATEMENT.

The work presented in this thesis is entirely my own. In Chapter IV, I have made use of some published results and their references are included wherever appropriate.

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LIST OF NOTATIONS

The following is the list of notations which, although standard, is not defined in the thesis:

$S = \{a, b, \ldots\}$	the set S consisting of $a, b, \ldots$
SUS^{-1}	the set consisting of $a, b, \ldots$ and $a^{-1}, b^{-1}, \ldots$
$S \setminus T$	the set of elements in S but not in the set T
$gp\{a, b, \ldots\}$	the group generated by $a, b, \ldots$
$\{a\}$	the cyclic group generated by a
J^+	the set of all positive integers
$J^+ \times J^+$	the set of all pairs (r,s), where $r, s \in J^+$
$\ker \theta$	the kernel of θ
ϕ	the empty set

CONTENTS

INTRODUCTION

It is a well-known theorem of W. Magnus [14] that if N is a normal subgroup of a free group F and if N' is the derived group of N, then the factor group F/N' is isomorphic to a certain group of 2×2 matrices.

D.E. Cohen has used Magnus's representation successfully to prove that metabelian varieties are finitely based (see for instance [15], Chapter 3). S. Bachmuth [1], [2] has investigated the automorphism group of a free metabelian group of finite rank by using the Magnus's representation. Recently, S. Bachmuth and I. Hughes [3] have used Magnus's representation to present simplified proofs of certain well-known theorems, (also see [4] for another application).

Apart from the applications, the problem of finding a faithful matrix representation of any class of groups seems to be of interest in itself.

Let $\underline{\underline{V}}$ denote the variety defined by the laws

2.1.15 $\quad [x,y;u,v;z]$ and

2.1.16 $\quad [x^{-1},y^{-1};u,v][x^{-1},v^{-1};y,u][x^{-1},u^{-1};v,y].$

$$.[v^{-1},y^{-1};x,u][y^{-1},u^{-1};x,v][u^{-1},v^{-1};x,y].$$

Let $V = V(F)$ denote the corresponding verbal subgroup of the free group F.

The main theorem in this thesis is that F/V is isomorphic to a certain group of 3×3 matrices. The example in Chapter III, shows that $\underline{\underline{V}}$ is a proper sub-variety of the variety of centre-extended-by-metabelian groups. Moreover, it follows from the proof of the main theorem that the above matrix representation is faithful for the free centre-extended-by-metabelian group of rank 3.

The above investigation was motivated by an altogether different problem which I now describe. Consider finitely generated groups of the variety of metabelian groups and of the variety of centre-extended-by-metabelian groups.

There are only a countable number of non-isomorphic finitely generated metabelian groups; whereas, there are uncountable infinity of non-isomorphic finitely generated centre-extended-by-metabelian groups (P. Hall [10]).

Finitely generated metabelian groups always satisfy the maximal condition for normal subgroups (P. Hall [10]); whereas, there exist finitely generated centre-extended-by-metabelian groups which do not satisfy the maximal condition for normal subgroups (P. Hall [12], also see M.F. Newman [16] for a 2-generator group).

Finitely generated metabelian groups are always residually finite; whereas, there exist finitely generated centre-extended-by-metabelian groups which are not residually finite (P. Hall [11]).

Thus, there is a wide gap between these two varieties of groups. The problem is to consider varieties inbetween these two varieties by imposing further suitable laws and to see in these new varieties the effect on the above properties. Although the faithful matrix representation mentioned earlier did not contribute directly to this problem, in obtaining it I learned more about the commutator structure of centre-extended-by-metabelian groups and was able to prove that a finitely generated centre-extended-by-metabelian group all of whose 2-generator subgroups are metabelian, satisfies the maximal condition for normal subgroups; and that a finitely generated centre-extended-by-metabelian group with the law $[x,y,y^2]$, satisfies the maximal condition for normal subgroups and is residually finite.

There are four chapters in this thesis. Chapter I consists of notations and commutator identities which have been repeatedly used in the rest of the thesis. The main theorem is established in Chapter II. In Chapter III, I construct an example of a four generator centre-extended-by-metabelian group in which the law

$$[x^{-1},y^{-1};u,v][x^{-1},v^{-1};y,u][x^{-1},u^{-1};v,y].$$

$$.[v^{-1},y^{-1};x,u][y^{-1},u^{-1};x,v][u^{-1},v^{-1};x,y].$$

does not hold. In Chapter IV, I describe some of the properties of finitely generated centre-extended-by-metabelian groups.

CHAPTER I

COMMUTATOR IDENTITIES

1.1 NOTATIONS.

For two elements a and b of a multiplicative group H, I write a^b to mean $b^{-1}ab$. The commutator of a and b is defined as

$$[a,b] = a^{-1}b^{-1}ab;$$

and for $n > 2$,

$$[a_1,a_2,\ldots,a_n] = [[a_1,a_2,\ldots,a_{n-1}],a_n]$$

defines a left-normed Commutator of weight n. Also, I define

$$[a,ob] = a$$

and

$$[a,rb] = [a,(r-1)b,b] \quad \text{for} \quad r \geq 1.$$

As usual I shall write

$$[[a,b],[c,d]] = [a,b;c,d].$$

If A and B are subgroups of H, then $[A,B]$ denotes the subgroup of H generated by all Commutators $[a,b]$ where $a \in A$ and $b \in B$. In particular, $H' = [H,H]$ is the derived group of H and $H'' = [H',H']$ is the second derived group of H. If H'' is trivial then H is called metabelian and if $[H'',H]$ is trivial then H is called centre-extended-by-metabelian.

1.2 COMMUTATOR IDENTITIES FOR GROUPS IN GENERAL.

The following commutator identities are well-known or can be easily verified.

1.2.1 $\quad [a,b] = [b,a]^{-1}$

1.2.2 $\quad a^b = a[a,b]$

1.2.3 $\quad [a,b]^c = [a^c,b^c]$

1.2.4 $\quad [ab,c] = [a,c]^b[b,c]$

1.2.5 $\quad [a,bc] = [a,c][a,b]^c$

1.2.6 $\quad [a,b] = [b^{-1},a]^b = [b,a^{-1}]^a$

1.2.7 $\quad [a,b,c^a][c,a,b^c][b,c,a^b] = 1$

1.3 COMMUTATOR IDENTITIES FOR METABELIAN GROUPS.

Let H be a metabelian group. If d is in H' and $a_1,a_2,\ldots,b_1,b_2,\ldots$ are in H then the following identities hold in H (see for instance [8]):

1.3.1 $\quad [d,a_1,\ldots,a_n] = [d^{-1},a_1,\ldots,a_n]^{-1} \qquad (n \geq 0).$

1.3.2 $\quad [d,a_1,\ldots,a_n] = [d,a_{1\sigma},\ldots,a_{n\sigma}] \qquad (n \geq 0)$

where σ is a premutation of $\{1,\ldots,n\}$.

1.3.3 $$[a_1,a_2,a_3,b_1,\ldots,b_r] = [a_1,a_3,a_2,b_1\ldots,b_r].$$

$$.\ [a_3,a_2,a_1,b_1,\ldots,b_r] \qquad (r \geq 0)$$

(The case $r = 0$ is the well-known Jacobi identity).

1.3.4 $$[a_1,\ldots,a_{s-1},a_s,b_1,\ldots,b_r] = [a_1,\ldots,a_{s-1},b_1,\ldots b_r]^{-1}.$$

$$.\ [a_1,\ldots,a_{s-2},a_s,b_1,\ldots,b_r]^{-1}.$$

$$.\ [a_1,\ldots,a_{s-2},a_{s-1}a_s,b_1,\ldots b_r] \qquad (r \geq 0,\ s \geq 3).$$

1.3.5 $$[a_1,\ldots,a_{s-1},a_{s-1}^{-1},b_1,\ldots,b_r] = [a_1,\ldots,a_{s-1},b_1,\ldots,b_r]^{-1}.$$

$$.\ [a_1,\ldots,a_{s-1}^{-1},b_1,\ldots,b_r]^{-1}$$

(This is the case $a_s = a_{s-1}^{-1}$ in 1.3.4).

1.4 COMMUTATOR IDENTITIES FOR CENTRE-EXTENDED-BY-METABELIAN GROUPS.

Let H be a centre-extended-by-metabelian group. If $d,d_1,d_2,\ldots$ are in H' and $a_1,a_2,\ldots,b,b_1,b_2,\ldots$ are in H then I list various identities which hold in H.

1.4.1 $$[d,d_1^k] = [d,d_1]^k$$

for all integers k.

Proof. If k is positive, then by 1.2.5

$$[d,d_1^k] = [d,d_1][d,d_1^{k-1}]^{d_1}$$

$$= [d,d_1][d,d_1^{k-1}],$$

and the result follows by induction on k.

If k is negative, then

$$[d,d_1^k] = [d,d_1^{-k}]^{-d_1^k} \quad \text{by 1.2.6, 1.2.1}$$

$$= [d,d_1^{-k}]^{-1}$$

$$= [d,d_1]^{(-k)(-1)} \quad \text{since } -k \text{ is positive,}$$

$$= [d,d_1]^k.$$

1.4.2 If $\prod_{i=1}^{n} d_i^{k_i} \in H''$, then $\prod_{i=1}^{n} [d,d_i]^{k_i} = 1$ for all integers k_i.

Proof.

$$1 = [d, \prod_{i=1}^{n} d_i^{k_i}]$$

$$= [d, (\prod_{i=1}^{n-1} d_i^{k_i})d_n^{k_n}]$$

$$= [d,d_n^{k_n}][d, \prod_{i=1}^{n-1} d_i^{k_i}] \quad \text{by 1.2.5}$$

$$= [d, \prod_{i=1}^{n-1} d_i^{k_i}][d,d_n]^{k_n} \quad \text{by 1.4.1}$$

and the proof follows by induction on n.

Using 1.3.1, 1.4.2 I get

1.4.3 $[d;d_1,a_1,\ldots,a_r]$

$$= [d;d_1^{-1},a_1,\ldots,a_r]^{-1} \qquad (r \geq 0);$$

using 1.3.2, 1.4.2 I get

1.4.4 $[d;d_1,a_1,\ldots,a_r]$

$$= [d;d_1,a_{1\sigma},\ldots,a_{r\sigma}] \qquad (r \geq 0);$$

and using 1.3.3, 1.4.2 I get

1.4.5 $[d;a_1,a_2,a_3,b_1,\ldots,b_r]$

$$= [d;a_1,a_3,a_2,b_1,\ldots,b_r].$$

$$.\ [d;a_3,a_2,a_1,b_1,\ldots,b_r] \qquad (r \geq 0).$$

Putting $a = d_2$, $b = d_1$ and $c = a_1$ in 1.2.7 and using 1.2.6 gives

$$[d_1,a_1,d_2] = [d_1^{a_1},[d_2,a_1^{-1}]^{a_1}]$$

$$= [d_1,[d_2,a_1^{-1}]]^{a_1} \qquad \text{by } 1.2.3$$

$$= [d_1,[d_2,a_1^{-1}]],$$

which on repeated application gives

1.4.6 $$[a_1,a_2,\dots,a_r;b_1,\dots,b_s]$$

$$= [a_1,a_2;b_1,\dots,b_s,a_r^{-1},\dots,a_3^{-1}] \qquad (r,s \geq 2).$$

Now, since

$$[d;a_1,a_2,a_2^{-1}]$$

$$= [d;a_1,a_2]^{-1}[d;a_1,a_2^{-1}]^{-1} \qquad \text{by } 1.3.5,\ 1.4.2,$$

replacing d by $[d,a_r^{-1},a_{r-1}^{-1},\dots,a_3^{-1}]$ and using 1.4.6 gives

1.4.7 $$[d;a_1,a_2,a_2^{-1},a_3,\dots,a_r]$$

$$= [d;a_1,a_2,a_3,\dots,a_r]^{-1}.$$

$$.\ [d;a_1,a_2^{-1},a_3,\dots,a_r]^{-1} \qquad (r \geq 2).$$

Next I prove

1.4.8 $$[a_1,b^{-1};a_2,b^{-1},b_1,\dots,b_s]$$

$$= [a_1,b;a_2,b,b_1,\dots,b_s] \qquad (s \geq 0)$$

<u>Proof</u>. Since

$$[a_1,b^{-1};a_2,b^{-1},b_1,\dots,b_s]$$

$$= [[b,a_1]^{b^{-1}};[b,a_2]^{b^{-1}},b_1,\dots,b_s] \qquad \text{by } 1.2.6$$

$$= [b,a_1;b,a_2,b_1^b,\dots,b_s^b] \qquad \text{by } 1.2.3$$

$$= [b,a_1;b,a_2,b_1[b_1,b],\dots,b_s[b_s,b]] \qquad \text{by } 1.2.2;$$

the result follows by repeated application of 1.2.5 and the fact that H is a centre-extended-by-metabelian group. $\bigcirc$

1.4.9 $$[a_1^{-1},a_2^{-1};a_3,a_4,a_5,\dots,a_{i-1},a_i,a_{i+1},\dots,a_r].$$
$$.[a_1^{-1},a_i^{-1};a_3,a_4,a_5,\dots,a_{i-1},a_{i+1},\dots,a_r,a_2]^{-1}.$$
$$.[a_3^{-1},a_2^{-1};a_1,a_4,a_5,\dots,a_{i-1},a_i,a_{i+1},\dots,a_r]^{-1}.$$
$$.[a_3^{-1},a_i^{-1};a_1,a_4,a_5,\dots,a_{i-1},a_{i+1},\dots,a_r,a_2]$$
$$= [a_2^{-1},a_i^{-1};a_1,a_3,a_4,a_5,\dots,a_{i-1},a_{i+1},\dots,a_r] \qquad (r \geq 4)$$

<u>Proof</u>.

$$[a_1^{-1},a_2^{-1};a_3,a_4,\dots,a_r]$$
$$= [a_1^{-1},a_2^{-1},a_i^{-1};a_3,a_4,\dots,a_{i-1},a_{i+1},\dots,a_r] \qquad \text{by } 1.4.4,\ 1.4.6$$
$$= [a_1^{-1},a_i^{-1};a_3,a_4,\dots,a_{i-1},a_{i+1},\dots,a_r,a_2].$$
$$.[a_i^{-1},a_2^{-1};a_3,a_4,\dots,a_{i-1},a_{i+1},\dots,a_r,a_1] \qquad \text{by } 1.4.5,\ 1.4.6$$
$$= [a_1^{-1},a_i^{-1};a_3,a_4,\dots,a_{i-1},a_{i+1},\dots,a_r,a_2].$$
$$.[a_i^{-1},a_2^{-1};a_3,a_1,a_4,\dots,a_{i-1},a_{i+1},\dots,a_r].$$
$$.[a_i^{-1},a_2^{-1};a_1,a_4,a_3,\dots,a_{i-1},a_{i+1},\dots,a_r] \qquad \text{by } 1.4.4,\ 1.4.5$$

$$= [a_1^{-1}, a_i^{-1}; a_3, a_4, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r, a_2].$$

$$.[a_2^{-1}, a_i^{-1}; a_1, a_3, a_4, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r].$$

$$.[a_3^{-1}, a_i^{-1}; a_1, a_4, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r, a_2]^{-1}.$$

$$.[a_3^{-1}, a_2^{-1}; a_1, a_4, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_r]$$

by 1.4.3, 1.4.4, 1.4.6 and 1.4.5;

which gives the required result. ○

1.4.10 $[a_1^{-1}, a_2^{-1}; a_3, a_4, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_r].$

$$.[a_2^{-1}, a_i^{-1}; a_3, a_4, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r, a_1]$$

$$= [a_1^{-1}, a_i^{-1}; a_3, a_4, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r, a_2]$$

<u>Proof</u>. The proof is contained in the proof of 1.4.9. ○

1.4.11 $[a^{-1}, c^{-1}; a^{-1}, b^{-1}]^{-1} [a^{-1}, b^{-1}; b^{-1}, c]^{-1} [c^{-1}, b^{-1}; b^{-1}, a^{-1}]$

$$[a^{-1}, b^{-1}; a^{-1}, c][a^{-1}, b; a^{-1}, c][c^{-1}, b^{-1}; b^{-1}, a]$$

$$[a^{-1}, c^{-1}; a^{-1}, b]^{-1} [c, b^{-1}; b^{-1}, a] = 1$$

Proof. Since,

$$[a^{-1},b^{-1};a^{-1},c][a^{-1},b;a^{-1},c][a^{-1},c^{-1};a^{-1},b]^{-1}[a^{-1},c^{-1};a^{-1},b^{-1}]^{-1}$$

$$= [a^{-1},b,b^{-1};a^{-1},c]^{-1}[a^{-1},c^{-1};a^{-1},b,b^{-1}] \qquad \text{by } 1.4.7$$

$$= [a^{-1},c;a^{-1},b,b^{-1}][a^{-1},c^{-1};a^{-1},b,b^{-1}] \qquad \text{by } 1.2.1$$

$$= [a^{-1},c,c^{-1};a^{-1},b,b^{-1}]^{-1} \qquad \text{by } 1.4.7$$

$$= [a,c,c^{-1};a,b,b^{-1}]^{-1} \qquad \text{by } 1.4.8$$

$$= [a,c,b;a,b,c]^{-1} \qquad \text{by } 1.4.6;$$

and

$$[c^{-1},b^{-1};b^{-1},a][a^{-1},b^{-1};b^{-1},c]^{-1}[c^{-1},b^{-1};b^{-1},a^{-1}][c,b^{-1};b^{-1},a]$$

$$= [b^{-1},c^{-1};b^{-1},a]^{-1}[b^{-1},a^{-1};b^{-1},c][b^{-1},c^{-1};b^{-1},a^{-1}]^{-1}[b^{-1},a;b^{-1},c]$$

$$\text{by } 1.4.3,\ 1.2.1$$

$$= [b^{-1},c^{-1};b^{-1},a,a^{-1}][b^{-1},a,a^{-1};b^{-1},c]^{-1} \qquad \text{by } 1.4.7$$

$$= [b^{-1},c^{-1};b^{-1},a,a^{-1}][b^{-1},c;b^{-1},a,a^{-1}] \qquad \text{by } 1.2.1$$

$$= [b^{-1},c,c^{-1};b^{-1},a,a^{-1}]^{-1} \qquad \text{by } 1.4.7$$

$$= [b,c,a;b,a,c]^{-1} \qquad \text{by } 1.4.8,\ 1.4.6;$$

and

$$[a,c,b;a,b,c]^{-1}[b,c,a;b,a,c]^{-1} = 1 \qquad \text{by } 1.4.5,\ 1.4.3,$$

the result follows.

CHAPTER II

A FAITHFUL MATRIX REPRESENTATION

In this chapter, I shall state and prove the main theorem. The preliminaries leading up to the statement of the main theorem are written in Section 2.1, and the proof of the main theorem is carried out in Sections 2.2, 2.3 and 2.4 of this chapter

SECTION 2.1

Let F be the free group freely generated by a set

$$X = \{x_1, x_2, \ldots\},$$

and let G be the free abelian group freely generated by a set

$$\underset{\sim}{X} = \{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots\}.$$

There is a homomorphism ξ of F onto G which maps x_i to $\underset{\sim}{x}_i$ for all i. If w is a word in F, then I denote its image under ξ by $\underset{\sim}{w}$; and in particular the identity e of F is mapped to the identity $\underset{\sim}{e}$ of G.

Let J denote the ring of all integers. The formal sums

$$j_1\underset{\sim}{w}_1 + j_2\underset{\sim}{w}_2 + \ldots + j_n\underset{\sim}{w}_n,$$

$j_i \in J$, $\underset{\sim}{w}_i \in G$ $(i = 1, \ldots, n)$, where addition and multiplication are defined in the obvious way, form a ring, the group ring of G over J, which I denote by JG. In JG the element 0 of J is identified with $0\underset{\sim}{w}$ of JG for all $\underset{\sim}{w}$ in G. Further I identify each element

j of J with the element $j\underset{\sim}{e}$ of JG and each word $\underset{\sim}{w}$ with the element $1.\underset{\sim}{w}$ of JG, so that J and G can be regarded as subsets of JG.

Let J^+ denote the set of all positive integers and let R denote the ring of all mappings of $J^+ \times J^+$ into JG with addition and multiplication defined as follows:

$$(\rho_1 + \rho_2)(r,s) = \rho_1(r,s) + \rho_2(r,s)$$

and $(\rho_1.\rho_2)(r,s) = \rho_1(r,s).\rho_2(r,s)$ for all $\rho_1,\rho_2 \in R$ and all $(r,s) \in J^+ \times J^+$. In R, I consider mappings λ_i, μ_j and $\nu_{\underset{\sim}{w}}$ defined as

2.1.1 $$\lambda_i(r,s) = \underset{\sim}{e} \quad \text{if} \quad r = i$$
$$= 0 \quad \text{if} \quad r \neq i;$$

2.1.2 $$\mu_j(r,s) = \underset{\sim}{e} \quad \text{if} \quad s = j$$
$$= 0 \quad \text{if} \quad s \neq j; \quad \text{and}$$

2.1.3 $$\nu_{\underset{\sim}{w}}(r,s) = \underset{\sim}{w} \quad \text{always.}$$

Let θ be the mapping of G into R defined as

$$\theta(\underset{\sim}{w}) = \nu_{\underset{\sim}{w}} \, .$$

Then it is easily seen that θ is an embedding of G into the multiplicative semigroup of R. Thus, I can identify $\nu_{\underset{\sim}{w}}$ with $\underset{\sim}{w}$ without any confusion.

Let $w = x_{i_1}^{\varepsilon_{i_1}} \ldots x_{i_\ell}^{\varepsilon_{i_\ell}} (\ell \geq 0,\ \varepsilon_{i_1}, \ldots, \varepsilon_{i_\ell} \in \{1,-1\})$

be an arbitrary word in F. I define mappings β, α and γ of F into R as follows:

2.1.4 $$\beta(w) = \sum_{j=1}^{\ell} \varepsilon_{i_j} \nu(\underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}-1/2} \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \ldots \underset{\sim}{x}_{i_\ell}^{\varepsilon_{i_\ell}})\lambda_{i_j}$$

$$= \sum_{j=1}^{\ell} \varepsilon_{i_j} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}-1/2} \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \ldots \underset{\sim}{x}_{i_\ell}^{\varepsilon_{i_\ell}} \lambda_{i_j};$$

2.1.5 $$\alpha(w) = \sum_{j=1}^{\ell} \varepsilon_{i_j} \underset{\sim}{x}_{i_1}^{\varepsilon_{i_1}} \underset{\sim}{x}_{i_2}^{\varepsilon_{i_2}} \ldots \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}-1/2} \mu_{i_j}; \text{ and}$$

2.1.6 $$\gamma(w) = \sum_{j=1}^{\ell} \frac{1-\varepsilon_{i_j}}{2} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}} \lambda_{i_j} \mu_{i_j}$$

$$+ \sum_{1 \leq j < k \leq \ell} \varepsilon_{i_j} \varepsilon_{i_k} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}-1/2} \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \ldots$$

$$\ldots \underset{\sim}{x}_{i_{k-1}}^{\varepsilon_{i_{k-1}}} \underset{\sim}{x}_{i_k}^{\varepsilon_{i_k}-1/2} \lambda_{i_j} \mu_{i_k}.$$

In particular, I have

2.1.7 (i) $\beta(e) = 0$, $\alpha(e) = 0$ and $\gamma(e) = 0$;

(ii) $\beta(x_i) = \lambda_i$, $\alpha(x_i) = \mu_i$ and $\gamma(x_i) = 0$ for all i.

And more generally, I prove

2.1.8 If w_1, w_2 are arbitrary words in F then

(i) $\beta(w_1\ w_2) = \underset{\sim}{w}_2\beta(w_1) + \beta(w_2)$

(ii) $\alpha(w_1\ w_2) = \alpha(w_1) + \underset{\sim}{w}_1\alpha(w_2)$

(iii) $\gamma(w_1\ w_2) = \gamma(w_1) + \gamma(w_2) + \beta(w_1)\alpha(w_2)$.

<u>Proof</u>. Let

$$w_1 = x_{i_1}^{\varepsilon_{i_1}} \dots x_{i_m}^{\varepsilon_{i_m}} \text{ and } w_2 = x_{i_{m+1}}^{\varepsilon_{i_{m+1}}} \dots x_{i_{m+n}}^{\varepsilon_{i_{m+n}}}.$$

Then

$$\beta(w_1\ w_2) = \sum_{j=1}^{m} \varepsilon_{i_j} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j} - 1/2}\, \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \dots \underset{\sim}{x}_{i_m}^{\varepsilon_{i_m}}\, \underset{\sim}{x}_{i_{m+1}}^{\varepsilon_{i_{m+1}}} \dots \underset{\sim}{x}_{i_{m+n}}^{\varepsilon_{i_{m+n}}} \lambda_{i_j}$$

$$+ \sum_{j=m+1}^{m+n} \varepsilon_{i_j} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j} - 1/2}\, \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \dots \underset{\sim}{x}_{i_{m+n}}^{\varepsilon_{i_{m+n}}} \lambda_{i_j} \quad \text{by } 2.1.4$$

$$= \underset{\sim}{x}_{i_{m+1}}^{\varepsilon_{i_{m+1}}} \cdots \underset{\sim}{x}_{i_{m+n}}^{\varepsilon_{i_{m+n}}} \left(\sum_{j=1}^{m} \varepsilon_{i_j} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}-1/2} \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \cdots \underset{\sim}{x}_{i_m}^{\varepsilon_{i_m}} \lambda_{i_j} \right)$$

$$+ \sum_{j=m+1}^{m+n} \varepsilon_{i_j} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}-1/2} \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \cdots \underset{\sim}{x}_{i_{m+n}}^{\varepsilon_{i_{m+n}}} \lambda_{i_j}$$

$$= \underset{\sim}{w}_2 \beta(w_1) + \beta(w_2) \quad \text{by } 2.1.4.$$

Similarly $\alpha(w_1\ w_2) = \alpha(w_1) + \underset{\sim}{w}_1 \alpha(w_2)$.

Finally

$$\gamma(w_1 w_2) = \sum_{j=1}^{m} \frac{1-\varepsilon_{i_j}}{2} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}} \lambda_{i_j} \mu_{i_j}$$

$$+ \sum_{j=m+1}^{m+n} \frac{1-\varepsilon_{i_j}}{2} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}} \lambda_{i_j} \mu_{i_j}$$

$$+ \sum_{1 \leq j < k \leq m} \varepsilon_{i_j} \varepsilon_{i_k} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}-1/2} \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \cdots$$

$$\cdots \underset{\sim}{x}_{i_{k-1}}^{\varepsilon_{i_{k-1}}} \underset{\sim}{x}_{i_k}^{\varepsilon_{i_k}-1/2} \lambda_{i_j} \mu_{i_k}$$

$$+ \sum_{m+1 \leq j < k \leq m+n} \varepsilon_{i_j} \varepsilon_{i_k} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}-1/2} \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \cdots$$

$$\cdots \underset{\sim}{x}_{i_{k-1}}^{\varepsilon_{i_{k-1}}} \underset{\sim}{x}_{i_k}^{\varepsilon_{i_k}-1/2} \lambda_{i_j} \mu_{i_k}$$

$$+ \sum_{\substack{1 \le j \le m, \\ m+1 \le k \le m+n}} \varepsilon_{i_j} \varepsilon_{i_k} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j} - 1/2} \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \cdots \underset{\sim}{x}_{i_{k-1}}^{\varepsilon_{i_{k-1}}} \underset{\sim}{x}_{i_k}^{\varepsilon_{i_k} - 1/2} \lambda_{i_j} \mu_{i_k}$$

by 2.1.6

$$= \gamma(w_1) + \gamma(w_2)$$

$$+ \sum_{\substack{1 \le j \le m, \\ m+1 \le k \le m+n}} \varepsilon_{i_j} \varepsilon_{i_k} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j} - 1/2} \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \cdots$$

$$\cdots \underset{\sim}{x}_{i_{k-1}}^{\varepsilon_{i_{k-1}}} \underset{\sim}{x}_{i_k}^{\varepsilon_{i_k} - 1/2} \lambda_{i_j} \mu_{i_k}$$

$$= \gamma(w_1) + \gamma(w_2)$$

$$+ \left(\sum_{j=1}^{m} \varepsilon_{i_j} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j} - 1/2} \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \cdots \underset{\sim}{x}_{i_{m-1}}^{\varepsilon_{i_{m-1}}} \underset{\sim}{x}_{i_m}^{\varepsilon_{i_m}} \lambda_{i_j} \right) .$$

$$\cdot \left(\sum_{k=m+1}^{m+n} \varepsilon_{i_k} \underset{\sim}{x}_{i_{m+1}}^{\varepsilon_{i_{m+1}}} \cdots \underset{\sim}{x}_{i_{k-1}}^{\varepsilon_{i_{k-1}}} \underset{\sim}{x}_{i_k}^{\varepsilon_{i_k} - 1/2} \mu_{i_k} \right)$$

$$= \gamma(w_1) + \gamma(w_2) + \beta(w_1)\alpha(w_2)$$ by 2.1.4, 2.1.5 .

Now 2.1.8 gives in particular

2.1.9 If w is an arbitrary word in F then

(i) $\beta(w^{-1}) = -\underset{\sim}{w}^{-1}\beta(w)$

(ii) $\alpha(w^{-1}) = -\underset{\sim}{w}^{-1}\alpha(w)$

(iii) $\gamma(w^{-1}) = -\gamma(w) + \underset{\sim}{w}^{-1}\beta(w)\alpha(w)$.

Let $w = x_{i_1}^{\varepsilon_{i_1}} \ldots x_{i_\ell}^{\varepsilon_{i_\ell}}$ $(\ell \geq 0)$ be a word in F and let $s,r,p,q \in J^+$. Let $\psi_s(w)$, $\Phi_r(w)$ and $\chi_{pq}(w)$ denote respectively the coefficient of λ_s, μ_r and $\lambda_p\mu_q$ in $\beta(w)$, $\alpha(w)$ and $\gamma(w)$. Then

2.1.10 $$\psi_s(w) = \sum_{i_j=s} \varepsilon_{i_j} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}-1/2}\, \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \ldots \underset{\sim}{x}_{i_\ell}^{\varepsilon_{i_\ell}} \qquad \text{by } 2.1.4;$$

2.1.11 $$\Phi_r(w) = \sum_{i_k=r} \varepsilon_{i_k} \underset{\sim}{x}_{i_1}^{\varepsilon_{i_1}}\, \underset{\sim}{x}_{i_2}^{\varepsilon_{i_2}} \ldots \underset{\sim}{x}_{i_k}^{\varepsilon_{i_k}-1/2} \qquad \text{by } 2.1.5;$$

and

2.1.12 $$\chi_{\substack{pq\\ p\neq q}}(w) = \sum_{(i_j,i_k)=(p,q)} \varepsilon_{i_j}\varepsilon_{i_k} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}-1/2}\, \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \ldots \underset{\sim}{x}_{i_{k-1}}^{\varepsilon_{i_{k-1}}}\, \underset{\sim}{x}_{i_k}^{\varepsilon_{i_k}-1/2},$$

$$\chi_{pp}(w) = \sum_{(i_j,i_k)=(p,p)} \varepsilon_{i_j}\varepsilon_{i_k} \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}-1/2}\, \underset{\sim}{x}_{i_{j+1}}^{\varepsilon_{i_{j+1}}} \ldots \underset{\sim}{x}_{i_{k-1}}^{\varepsilon_{i_{k-1}}}\, \underset{\sim}{x}_{i_k}^{\varepsilon_{i_k}-1/2}$$

$$+ \sum_{i_j=p} \frac{1-\varepsilon_{i_j}}{2}\, \underset{\sim}{x}_{i_j}^{\varepsilon_{i_j}} \qquad \text{by } 2.1.6.$$

Thus, if $m = \max\{i_1, \ldots, i_l\}$ then

2.1.13 (i) $\beta(w) = \sum_{s=1}^{m} \psi_s(w)\lambda_s$

(ii) $\alpha(w) = \sum_{r=1}^{m} \Phi_r(w)\mu_r$

(iii) $\gamma(w) = \sum_{\substack{1 \le p \le m \\ 1 \le q \le m}} x_{pq}(w)\lambda_p\mu_q$.

As a consequence of 2.1.13. I obtain (by using 2.1.1, 2.1.2)

2.1.14 (i) If $\beta(w) = 0$, then $\psi_s(w) = 0$ for all $s \in J^+$.

(ii) If $\alpha(w) = 0$, then $\Phi_r(w) = 0$ for all $r \in J^+$.

(iii) If $\gamma(w) = 0$, then $x_{pq}(w) = 0$ for all $p,q \in J^+$.

Let M^* denote the ring of all 3 x 3 matrices of the form

$$\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix}$$

with entries in R. Then the subset M of M^* consisting of all matrices of the form

$$<w> = \begin{pmatrix} \underset{\sim}{e} & 0 & 0 \\ \alpha(w) & \underset{\sim}{w} & 0 \\ \gamma(w) & \beta(w) & \underset{\sim}{e} \end{pmatrix} \quad 1)$$

(where 0 is the mapping which maps each (r,s) to 0 in JG) is a multiplicative subgroup of M^*; since

$$<w_1><w_2^{-1}> = \begin{pmatrix} \underset{\sim}{e} & 0 & 0 \\ \alpha(w_1) & \underset{\sim}{w}_1 & 0 \\ \gamma(w_1) & \beta(w_1) & \underset{\sim}{e} \end{pmatrix} \begin{pmatrix} \underset{\sim}{e} & 0 & 0 \\ \alpha(w_2^{-1}) & \underset{\sim}{w}_2^{-1} & 0 \\ \gamma(w_2^{-1} & \beta(w_2^{-1}) & \underset{\sim}{e} \end{pmatrix}$$

$$= \begin{pmatrix} \underset{\sim}{e} & 0 & 0 \\ \alpha(w_1) + \underset{\sim}{w}_1\alpha(w_2^{-1}) & \underset{\sim}{w}_1\underset{\sim}{w}_2^{-1} & 0 \\ \gamma(w_1) + \beta(w_1)\alpha(w_2^{-1}) + \gamma(w_2^{-1}) & \underset{\sim}{w}_2^{-1}\beta(w_1) + \beta(w_2^{-1}) & \underset{\sim}{e} \end{pmatrix}$$

---

1) It will be seen later that the correspondence $w \longrightarrow <w>$ is not one-to-one.

$$= \begin{pmatrix} \underset{\sim}{e} & 0 & 0 \\ \alpha(w_1 w_2^{-1}) & \underset{\sim}{w}_1 \underset{\sim}{w}_2^{-1} & 0 \\ \gamma(w_1 w_2^{-1}) & \beta(w_1 w_2^{-1}) & \underset{\sim}{e} \end{pmatrix}$$

by 2.1.8 .

Let $\underline{\underline{V}}$ denote the variety defined by the laws

2.1.15 $[x,y;u,v;z]$

and

2.1.16 $[x^{-1},y^{-1};u,v][x^{-1},v^{-1};y,u][x^{-1},u^{-1};v,y].$

$.[v^{-1},y^{-1};x,u][y^{-1},u^{-1};x,v][u^{-1},v^{-1};x,y].$

Let $V = V(F)$ denote the corresponding verbal subgroup of the free group F. There exists a centre-extended-by-metabelian group which does not satisfy the law 2.1.16 (see Chapter III). Thus the variety $\underline{\underline{V}}$ is a proper sub-variety of the variety of centre-extended-by-metabelian groups.

Now the main theorem can be stated as follows:

<u>THEOREM</u>. M is a free $\underline{\underline{V}}$-group freely generated by all matrices of the form

$$< x_i > \; = \begin{pmatrix} \tilde{e} & 0 & 0 \\ \mu_i & \tilde{x}_i & 0 \\ 0 & \lambda_i & \tilde{e} \end{pmatrix}.$$

As on page 20, it is easily seen that

$$< w_1 >< w_2 > \; = \; < w_1 w_2 > ,$$

so the mapping η of F into M defined by

$$\eta : w \longrightarrow < w >$$

is a homomorphism.

The proof of the theorem consists in showing that

$$\ker \eta = V.$$

The inclusion $V \subseteq \ker \eta$ is proved in Section 2.2. The reverse inclusion takes much longer and is proved in Sections 2.3 and 2.4. If $w \in \ker \eta$, then I show firstly (in Section 2.3) that $w \in F''$; and secondly (in Section 2.4) that $w \in V$.

SECTION 2.2

If d_1, d_2 are in F', then from 2.1.8(i) it follows that

$$\beta(d_1 d_2) = \beta(d_1) + \beta(d_2),$$

which together with 2.1.9(i) gives

$$\beta(d_1^{-1} d_2^{-1} d_1 d_2) = -\beta(d_1) - \beta(d_2) + \beta(d_1) + \beta(d_2)$$

$$= 0.$$

Thus

2.2.1 $$\beta([d_1, d_2]) = 0.$$

Using 2.1.8(ii), 2.1.9(ii), a similar argument gives

2.2.2 $$\alpha([d_1, d_2]) = 0$$

Further, since $[d_1, d_2] = (d_2 d_1)^{-1}(d_1 d_2)$;

$$\gamma([d_1, d_2]) = \gamma((d_2 d_1)^{-1}) + \gamma(d_1 d_2) + \beta((d_2 d_1)^{-1})\alpha(d_1 d_2)$$

by 2.1.8(iii)

$$= -\gamma(d_2 d_1) + \beta(d_2 d_1)\alpha(d_2 d_1) + \gamma(d_1 d_2) - \beta(d_2 d_1)\alpha(d_1 d_2)$$

by 2.1.9(i), (iii)

$$= -\gamma(d_2 d_1) + \gamma(d_1 d_2)$$

(since $\alpha(d_2 d_1) = \alpha(d_1 d_2 [d_2, d_1]) = \alpha(d_1 d_2)$ by 2.2.2), which gives

2.2.3 $\qquad \gamma([d_1, d_2]) = \beta(d_1)\alpha(d_2) - \beta(d_2)\alpha(d_1)$ by 2.1.8(iii).

In 2.2.3 take $d_1 = [x_1, x_2]$ and $d_2 = [x_3, x_4]$ where $x_1, x_2, x_3, x_4 \in X$. Then

$$\gamma([x_1, x_2; x_3, x_4]) = \beta([x_1, x_2])\alpha([x_3, x_4]) - \beta([x_3, x_4])\alpha([x_1, x_2]);$$

$$\begin{aligned} \beta([x_3, x_4]) &= \beta(x_3^{-1} x_4^{-1} x_3 x_4) \\ &= -\underset{\sim}{x}_3^{-1} \underset{\sim}{x}_4^{-1} \underset{\sim}{x}_3 \underset{\sim}{x}_4 \lambda_3 - \underset{\sim}{x}_4^{-1} \underset{\sim}{x}_3 \underset{\sim}{x}_4 \lambda_4 \\ &\quad + \underset{\sim}{x}_4 \lambda_3 + \lambda_4 \qquad \text{by } 2.1.4; \end{aligned}$$

$$\begin{aligned} \alpha([x_1, x_2]) &= \alpha(x_1^{-1} x_2^{-1} x_1 x_2) \\ &= -\underset{\sim}{x}_1^{-1} \mu_1 - \underset{\sim}{x}_1^{-1} \underset{\sim}{x}_2^{-1} \mu_2 \\ &\quad + \underset{\sim}{x}_1^{-1} \underset{\sim}{x}_2^{-1} \mu_1 + \underset{\sim}{x}_1^{-1} \underset{\sim}{x}_2^{-1} \underset{\sim}{x}_1 \mu_2 \qquad \text{by } 2.1.5; \end{aligned}$$

so that

$$\begin{aligned} \chi_{31}([x_1, x_2; x_3, x_4]) &= \underset{\sim}{x}_1^{-1} - \underset{\sim}{x}_1^{-1} \underset{\sim}{x}_2^{-1} - \underset{\sim}{x}_4 \underset{\sim}{x}_1^{-1} + \underset{\sim}{x}_1^{-1} \underset{\sim}{x}_2^{-1} \underset{\sim}{x}_4 \\ &\neq 0 \quad (\text{since } \underset{\sim}{x}_i \in \underset{\sim}{X}). \end{aligned}$$

Thus

$$\gamma([x_1,x_2;x_3,x_4]) \neq 0.$$

Further if $f \in F$, then

$$\begin{aligned}
\gamma([[d_1,d_2],f]) &= \gamma([d_2,d_1]f^{-1}[d_1,d_2]f) \\
&= \gamma([d_2,d_1]f^{-1}) + \gamma([d_1,d_2]f) \\
&\quad + \beta([d_2,d_1]f^{-1})\alpha([d_1,d_2]f) \qquad \text{by } 2.1.8\text{(iii)} \\
&= \gamma([d_2,d_1]) + \gamma(f^{-1}) + \gamma([d_1,d_2]) \\
&\quad + \gamma(f) + \beta(f^{-1})\alpha(f) \\
&\qquad\qquad \text{by } 2.1.8,\ 2.1.9,\ 2.2.1 \text{ and } 2.2.2 \\
&= \underset{\sim}{f}^{-1}\beta(f)\alpha(f) - \underset{\sim}{f}^{-1}\beta(f)\alpha(f) \\
&\qquad\qquad \text{by } 2.1.9\text{(i), (iii), } 2.2.1,\ 2.2.2 \\
&= 0,
\end{aligned}$$

so that

2.2.4 $$\gamma([[d_1,d_2],f]) = 0.$$

Let f_1, f_2, f_3, f_4 be elements in F and let

$$
\begin{aligned}
f^* \;=\;& [f_1^{-1}, f_2^{-1}; f_3, f_4][f_1^{-1}, f_4^{-1}; f_2, f_3]. \\
& .[f_1^{-1}, f_3^{-1}; f_4, f_2][f_4^{-1}, f_2^{-1}; f_1, f_3]. \\
& .[f_2^{-1}, f_3^{-1}; f_1, f_4][f_3^{-1}, f_4^{-1}; f_1, f_2] \;.
\end{aligned}
$$

Then

$$
\begin{aligned}
\gamma(f^*) \;=\;& \gamma([f_1^{-1}, f_2^{-1}; f_3, f_4]) + \gamma([f_1^{-1}, f_4^{-1}; f_2, f_3]) \\
& + \gamma([f_1^{-1}, f_3^{-1}; f_4, f_2]) + \gamma([f_4^{-1}, f_2^{-1}; f_1, f_3]) \\
& + \gamma([f_2^{-1}, f_3^{-1}; f_1, f_4]) + \gamma([f_3^{-1}, f_4^{-1}; f_1, f_2])
\end{aligned}
$$

by 2.1.8(iii), 2.2.1 and 2.2.2.

By 2.2.3, 2.1.8(i), (ii)

$$
\begin{aligned}
\gamma([f_1^{-1}, f_2^{-1}; f_3, f_4]) =\;& \Big(\underset{\sim}{f}_1^{-1}\beta(f_1) + \underset{\sim}{f}_1^{-1}\underset{\sim}{f}_2^{-1}\beta(f_2) + \underset{\sim}{f}_2^{-1}\beta(f_1^{-1}) + \beta(f_2^{-1})\Big). \\
& .\Big(\alpha(f_3^{-1}) + \underset{\sim}{f}_3^{-1}\alpha(f_4^{-1}) + \underset{\sim}{f}_3^{-1}\underset{\sim}{f}_4^{-1}\alpha(f_3) + \underset{\sim}{f}_4^{-1}\alpha(f_4)\Big) \\
& - \Big(\underset{\sim}{f}_3\beta(f_3^{-1}) + \underset{\sim}{f}_3\underset{\sim}{f}_4\beta(f_4^{-1}) + \underset{\sim}{f}_4\beta(f_3) + \beta(f_4)\Big). \\
& .\Big(\alpha(f_1) + \underset{\sim}{f}_1\alpha(f_2) + \underset{\sim}{f}_1\underset{\sim}{f}_2\alpha(f_1^{-1}) + \underset{\sim}{f}_2\alpha(f_2^{-1})\Big) \\
=\;& \Big(\underset{\sim}{f}_1^{-1}\beta(f_1) + \underset{\sim}{f}_1^{-1}\underset{\sim}{f}_2^{-1}\beta(f_2) - \underset{\sim}{f}_1^{-1}\underset{\sim}{f}_2^{-1}\beta(f_1) - \underset{\sim}{f}_2^{-1}\beta(f_2)\Big). \\
& .\Big(-\underset{\sim}{f}_3^{-1}\alpha(f_3) - \underset{\sim}{f}_3^{-1}\underset{\sim}{f}_4^{-1}\alpha(f_4) + \underset{\sim}{f}_3^{-1}\underset{\sim}{f}_4^{-1}\alpha(f_3) + \underset{\sim}{f}_4^{-1}\alpha(f_4)\Big) \\
& - \Big(-\underset{\sim}{f}_3\underset{\sim}{f}_3^{-1}\beta(f_3) - \underset{\sim}{f}_3\underset{\sim}{f}_4\underset{\sim}{f}_4^{-1}\beta(f_4) + \underset{\sim}{f}_4\beta(f_3) + \beta(f_4)\Big). \\
& .\Big(\alpha(f_1) + \underset{\sim}{f}_1\alpha(f_2) - \underset{\sim}{f}_1^{-1}\underset{\sim}{f}_1\underset{\sim}{f}_2\alpha(f_1) - \underset{\sim}{f}_2\underset{\sim}{f}_2^{-1}\alpha(f_2)\Big)
\end{aligned}
$$

by 2.1.9(i), (ii);

and writing out the corresponding expressions for the other five terms in the same way and adding all of them gives that $\gamma(f^*) = 0$. Thus $M \in \underline{\underline{V}}$. This completes the proof that $V \subseteq \ker \eta$.

SECTION 2.3

Let N be a normal subgroup of the free group F and N' be the derived group of N. Let $t_1, t_2, \ldots$ be indeterminates which commute with all elements of the group-ring $J(F/N)$. Then a theorem of Magnus [14] [2)] states:

the mapping $\bar{\eta}$ of F/N' into 2×2 matrices defined by

$$\bar{\eta} : x_i N' \longrightarrow \begin{pmatrix} x_i N & 0 \\ t_i & 1 \end{pmatrix}$$

is an isomorphism.

In the above theorem, if $N = F'$ then it follows (in my notation):

If $\bar{\eta}$ is the homomorphism of F into 2×2 matrices over R defined by

$$\bar{\eta} : x_i \longrightarrow \begin{pmatrix} \underset{\sim}{x}_i & 0 \\ \lambda_i & \underset{\sim}{e} \end{pmatrix},$$

then $\ker \bar{\eta} = F''$.

---

2) Magnus proves this theorem only for the case when F/N is finite. A complete proof of this theorem is given by R.H. Fox [5].

This theorem immediately implies in my case that $w \in F''$, since $\beta(w) = 0$. However, in this section, I shall give an independent proof that $w \in F''$.

Let w be an arbitrary word in F such that $w \in \ker \eta$; then $\alpha(w) = 0$, $\beta(w) = 0$ and $\gamma(w) = 0$. Since $\underset{\sim}{w} = \underset{\sim}{e}$, it follows that $w \in F'$ and so w can be written as

$$w = C_1^{\delta_1} C_2^{\delta_2} \dots C_n^{\delta_n} w_1'' ;$$

where $C_1, C_2, \dots, C_n$ are left-normed commutators with entries in $X \cup X^{-1}$, $\delta_i \in \{1,-1\}$ and $w_1'' \in F''$.

Let m,n be positive integers and let C be a left-normed commutator in F in which m entries are equal to u and n entries are equal to u^{-1} for some $u \in X \cup X^{-1}$. Then modulo F'', by 1.3.2 and/or 1.3.1, C can be written as $C'^{\pm 1}$, where in C' two of the entries u and u^{-1} are adjacent. Further, by 1.3.5, C' can be written as a product of two commutators; one containing m u's and $(n-1)$ u^{-1}'s and the other containing $(m-1)$ u's and n u^{-1}'s. By repeated application of this argument, C can be written as a power product [3] of commutators each containing u or u^{-1} but not both u and u^{-1}.

---

3) Here and throughout this Chapter by "power product" I shall mean a product of commutators and/or their inverses.

Thus, I can take each C_i to be of the form

$$C_i = [u_1, u_2, \dots, u_s] \quad (s \geq 2 \ , \ u_i \in XUX^{-1})$$

where

2.3.1 $$\{u_1, u_2, \dots, u_s\} \cap \{u_1^{-1}, u_2^{-1}, \dots, u_s^{-1}\} = \phi.$$

Now I arrange the factors of w in the order of decreasing weights that is

2.3.2 $$\mathrm{wt}C_1 \geq \mathrm{wt}C_2 \geq \dots \geq \mathrm{wt}C_n$$

$$(\mathrm{wt}C_i = \text{weight of } C_i).$$

Next I prove the following results:

2.3.3 If $w_1', w_2' \in F$, then

$$\beta([w_1', w_2']) = (-1 + \underset{\sim}{w_2'})\beta(w_1') - (-1 + \underset{\sim}{w_1'})\beta(w_2')$$

<u>Proof</u>.

$$\beta([w_1', w_2']) = \beta((w_2'w_1')^{-1}w_1'w_2')$$

$$= \underset{\sim}{w_1'}\underset{\sim}{w_2'}\beta((w_2'w_1')^{-1}) + \beta(w_1'w_2') \qquad \text{by } 2.1.8(i)$$

$$= -\beta(w_2'w_1') + \beta(w_1'w_2') \qquad \text{by } 2.1.9(i)$$

$$= (-1 + \underset{\sim}{w_2'})\beta(w_1') - (-1 + \underset{\sim}{w_1'})\beta(w_2') \quad \text{by } 2.1.8(i).$$

2.3.4 If $C = [x_i^{\varepsilon_i}, x_j^{\varepsilon_j}, u_1, \ldots, u_r]$ $(r \geq 0, \quad u_i \in XUX^{-1})$

then

(i) $$\beta(C) = (-1 + \underset{\sim}{u}_1) \ldots (-1 + \underset{\sim}{u}_r)\beta([x_i^{\varepsilon_i}, x_j^{\varepsilon_j}])$$

(ii) $$\beta(C) = \psi_i(C)\lambda_i + \psi_j(C)\lambda_j,$$ where

$$\psi_i(C) = \varepsilon_i \underset{\sim}{x}_i^{(\varepsilon_i - 1)/2}(-1 + \underset{\sim}{x}_j^{\varepsilon_j})(-1 + \underset{\sim}{u}_1) \ldots (-1 + \underset{\sim}{u}_r)$$

and

$$\psi_j(C) = -\varepsilon_j \underset{\sim}{x}_j^{(\varepsilon_j - 1)/2}(-1 + \underset{\sim}{x}_i^{\varepsilon_i})(-1 + \underset{\sim}{u}_1) \ldots (-1 + \underset{\sim}{u}_r).$$

<u>Proof</u>.

$$\beta([x_i^{\varepsilon_i}, x_j^{\varepsilon_j}, u_1, \ldots, u_{r-1}, u_r]) = (-1 + \underset{\sim}{u}_r)\beta([x_i^{\varepsilon_i}, x_j^{\varepsilon_j}, u_1, \ldots, u_{r-1}])$$

(by taking $w_1' = [x_i^{\varepsilon_i}, x_j^{\varepsilon_j}, u_1, \ldots, u_{r-1}]$, $w_2' = u_r$ in 2.3.3)

$$= (-1 + \underset{\sim}{u}_1) \ldots (-1 + \underset{\sim}{u}_r)\beta([x_i^{\varepsilon_i}, x_j^{\varepsilon_j}])$$

(by repeated application of the above argument).

Also, if $w_1' = x_i^{\varepsilon_i}$, $w_2' = x_j^{\varepsilon_j}$ in 2.3.3 then

$$\beta([x_i^{\varepsilon_i}, x_j^{\varepsilon_j}]) = (-1 + \underset{\sim}{x}_j^{\varepsilon_j})\beta(x_i^{\varepsilon_i}) - (-1 + \underset{\sim}{x}_i^{\varepsilon_i})\beta(x_j^{\varepsilon_j})$$

$$= \varepsilon_i \underset{\sim}{x}_i^{\varepsilon_i - 1/2}(-1 + \underset{\sim}{x}_j^{\varepsilon_j})\lambda_i$$

$$-\varepsilon_j \underset{\sim}{x}_j^{\varepsilon_j - 1/2}(-1 + \underset{\sim}{x}_i^{\varepsilon_i})\lambda_j \qquad \text{by} \quad 2.1.9(\text{i}).$$

Using this in 2.3.4(i), I get

$$\beta([x_i^{\varepsilon_i}, x_j^{\varepsilon_j}, u_1, \ldots, u_{r-1}, u_r])$$

$$= (-1 + \underset{\sim}{u}_1) \ldots (-1 + \underset{\sim}{u}_r) \left(\varepsilon_i \underset{\sim}{x}_i^{\varepsilon_i - 1/2}(-1 + \underset{\sim}{x}_j^{\varepsilon_j})\lambda_i \right.$$

$$\left. -\varepsilon_j \underset{\sim}{x}_j^{\varepsilon_j - 1/2}(-1 + \underset{\sim}{x}_i^{\varepsilon_i})\lambda_j \right)$$

$$= \varepsilon_i \underset{\sim}{x}_i^{\varepsilon_i - 1/2}(-1 + \underset{\sim}{x}_j^{\varepsilon_j})(-1 + \underset{\sim}{u}_1) \ldots (-1 + \underset{\sim}{u}_r)\lambda_i$$

$$-\varepsilon_j \underset{\sim}{x}_j^{\varepsilon_j - 1/2}(-1 + \underset{\sim}{x}_i^{\varepsilon_i})(-1 + \underset{\sim}{u}_1) \ldots (-1 + \underset{\sim}{u}_r)\lambda_j .$$

Now, I am in a position to prove that $w \in F''$. I take a minimal representation of w in the form

$$w = C_1^{\delta_1} C_2^{\delta_2} \ldots . \; C_n^{\delta_n} \, w'' \quad (n \geq 1)$$

where $\delta_i \in \{1, -1\}$, $w'' \in F''$ and each C_i satisfies 2.3.1, 2.3.2.

Write

$$C_1^{\delta_1} = [x_{i_1}^{\varepsilon_1}, x_{i_2}^{\varepsilon_2}, r_1 x_{i_1}^{\varepsilon_1}, r_2 x_{i_2}^{\varepsilon_2}, (r_3+1)x_{i_3}^{\varepsilon_3}, \ldots, (r_t+1)x_{i_t}^{\varepsilon_t}]^{\delta_1}$$

where $r_i \geq 0$, $t \geq 2$ and $i_j \neq i_k$ for $j \neq k$.

By 2.3.4, 2.1.9(i)

$$\psi_{i_1}(C_1^{\delta_1}) = \delta_1 \varepsilon_1 \underset{\sim}{x}_{i_1}^{\varepsilon_1 - 1/2}(-1 + \underset{\sim}{x}_{i_1}^{\varepsilon_1})^{r_1}(-1 + \underset{\sim}{x}_{i_2}^{\varepsilon_2})^{(r_2+1)} \ldots (-1 + \underset{\sim}{x}_{i_t}^{\varepsilon_t})^{(r_t+1)}.$$

A term of $\psi_{i_1}(C_1^{\delta_1})$ is $\delta_1 \varepsilon_1 \underset{\sim}{x}_{i_1}^{\varepsilon_1 - 1/2} \underset{\sim}{x}_{i_1}^{\varepsilon_1 r_1} \underset{\sim}{x}_{i_2}^{\varepsilon_2 (r_2+1)} \ldots \underset{\sim}{x}_{i_t}^{\varepsilon_t (r_t+1)}$.

Since C_1 satisfies 2.3.1, this term has maximum length precisely 4)

$r^* - 1 + \tau$, where $r^* = \sum_{k=1}^{t} (r_k+1)$ and $\tau = \begin{cases} 0 & \text{if } \varepsilon_1 = 1 \\ 1 & \text{if } \varepsilon_1 = -1 \end{cases}$;

and this is the only term of $\psi_{i_1}(C_1^{\delta_1})$ of length $r^* - 1 + \tau$.

By 2.1.8(i) and 2.2.1

$$\beta(w) = \beta(C_1^{\delta_1}) + \ldots + \beta(C_n^{\delta_n})$$

and by hypothesis $\beta(w) = 0$; thus it follows that

$$0 = \psi_{i_1}(w) = \psi_{i_1}(C_1^{\delta_1}) + \ldots + \psi_{i_1}(C_n^{\delta_n}).$$

---

4) Let $\underset{\sim}{w} = \underset{\sim}{u}'_1 \ldots \underset{\sim}{u}'_s$ be a word in G, if $\underset{\sim}{w}$ is reduced then I say that $\underset{\sim}{w}$ is of length precisely s.

Thus, for some $k \geq 1$, a term of $\psi_{i_1}(C_k^{\delta_k})$ is

$$-\delta_1 \varepsilon_1 \underset{\sim}{x}_{i_1}^{\varepsilon_1 - 1/2} \underset{\sim}{x}_{i_1}^{\varepsilon_1 r_1} \underset{\sim}{x}_{i_2}^{\varepsilon_2(r_2+1)} \cdots \underset{\sim}{x}_{i_t}^{\varepsilon_t(r_t+1)} .$$

Clearly $k \neq 1$, since $\psi_{i_1}(C_1^{\delta_1})$ has precisely one term of length $r^* - 1 + \tau$. Without loss of generality I can take $k = 2$ and assume that $\psi_{i_1}(C_2^{\delta_2})$ contains a term

$$-\delta_1 \varepsilon_1 \underset{\sim}{x}_{i_1}^{\varepsilon_1 - 1/2} \underset{\sim}{x}_{i_1}^{\varepsilon_1 r_1} \underset{\sim}{x}_{i_2}^{\varepsilon_2(r_2+1)} \cdots \underset{\sim}{x}_{i_t}^{\varepsilon_t(r_t+1)} .$$

By 2.3.4, 1.3.1 and 2.3.2 I can take

$$C_2^{\delta_2} = [x_{i_1}^{\varepsilon_1'}, v_1, \ldots, v_r]^{\delta_2} ,$$

where $v_1, \ldots, v_r \in XUX^{-1}$, $\varepsilon_1' \in \{1, -1\}$ and $r + 1 \leq r^*$.

By 2.3.4, 2.1.9(i)

$$\psi_{i_1}(C_2^{\delta_2}) = \delta_2 \varepsilon_1' \underset{\sim}{x}_{i_1}^{\varepsilon_1' - 1/2} (-1 + \underset{\sim}{v}_1) \cdots (-1 + \underset{\sim}{v}_r) ;$$

if $\varepsilon_1' = -\varepsilon_1$ then $\psi_{i_1}(C_2^{\delta_2})$ would not contain a term having

$\underset{\sim}{x}_{i_1}^{\frac{\varepsilon_1 - 1}{2} + r_1 \varepsilon_1}$ as a factor (because C_2 satisfies 2.3.1), thus

$\varepsilon_1' = \varepsilon_1$; and it follows that in the set $\{v_1, \ldots, v_r\}$ there are

$r_1x_{i_1}^{\varepsilon_1}, s$, $(r_2+1)x_{i_2}^{\varepsilon_2}, s, \ldots, (r_t+1)x_{i_t}^{\varepsilon_t}, s$ which gives

$$r \geq r^* - 1; \quad r+1 \geq r^*; \quad r+1 = r^* \quad (\text{since } r+1 \leq r^*);$$

now this implies that $\delta_2 = -\delta_1$. Therefore,

$$C_2^{\delta_2} = [x_{i_1}^{\varepsilon_1}, x_{i_{2\sigma}}^{\varepsilon_{2\sigma}}, r_1x_{i_1}^{\varepsilon_1}, r_{2\sigma}x_{i_{2\sigma}}^{\varepsilon_{2\sigma}}, (r_{3\sigma}+1)x_{i_{3\sigma}}^{\varepsilon_{3\sigma}}, \ldots, (r_{t\sigma}+1)x_{i_{t\sigma}}^{\varepsilon_{t\sigma}}]^{-\delta_1}$$

where σ is a permutation of $\{2,3,\ldots,t\}$; if $2\sigma = 2$ then

$C_1^{\delta_1}C_2^{\delta_2} = e$ modulo F'' by 1.3.2 —— a contradiction to the minimal representation of w;

If $2\sigma \neq 2$, then without any loss of generality I take $2\sigma = 3$ and $3\sigma = 2$. Then by 1.3.3

$$C_2^{\delta_2} = [x_{i_1}^{\varepsilon_1}, x_{i_2}^{\varepsilon_2}, r_1x_{i_1}^{\varepsilon_1}, r_2x_{i_2}^{\varepsilon_2}, (r_3+1)x_{i_3}^{\varepsilon_3}, \ldots, (r_t+1)x_{i_t}^{\varepsilon_t}]^{-\delta_1}\,.$$

$$.[x_{i_2}^{\varepsilon_2}, x_{i_3}^{\varepsilon_3}, (r_1+1)x_{i_1}^{\varepsilon_1}, r_2x_{i_2}^{\varepsilon_2}, r_3x_{i_3}^{\varepsilon_3}, \ldots, (r_t+1)x_{i_t}^{\varepsilon_t}]^{-\delta_1} \quad \text{modulo } F''$$

and

$$C_1^{\delta_1}C_2^{\delta_2} = [x_{i_3}^{\varepsilon_3}, x_{i_2}^{\varepsilon_2}, r_2x_{i_2}^{\varepsilon_2}, r_3x_{i_3}^{\varepsilon_3}, (r_1+1)x_{i_1}^{\varepsilon_1}, \ldots, (r_t+1)x_{i_t}^{\varepsilon_t}]^{\delta_1}$$

by 1.3.1 modulo F''

$$= C_{1_2}^{\delta_1} \quad (\text{say}) \,,$$

thus

$$w = C_{1_2}^{\delta_1} C_3^{\delta_3} \ldots C_n^{\delta_n} w_2''' \quad (\text{where } w_2''' \in F''),$$

which again contradicts the choice of w.

SECTION 2.4

In this section I prove that if w is represented by the unit matrix then $w \in V$.

Since $w \in F''$ (proved in section 2.3), it can be written as

$$w = C_1^{\delta_1} C_2^{\delta_2} \ldots C_m^{\delta_m} \hat{w} \quad (m \geq 1, \ \delta_i \in \{1,-1\}),$$

where each C_i is a commutator in $F'' \setminus [F'',F]$ with entries from XUX^{-1}, $\hat{w} \in [F'',F]$ and

2.4.1 $\qquad wtC_1 \geq wtC_2 \geq \ldots \geq wtC_m$

Now, I show that I can take each C_i to be of the form

2.4.2 $\qquad [u_1^{-1}, u_2^{-1}; u_3, u_4, \ldots, u_s] (s \geq 4, \ u_i \in XUX^{-1}),$

where

2.4.3 $\qquad \{u_3, u_4, \ldots, u_s\} \cap \{u_3^{-1}, u_4^{-1}, \ldots, u_s^{-1}\} = \emptyset,$

2.4.4 $\qquad \{u_1^{-1}, u_2^{-1}\} \cap \{u_5, \ldots, u_s\} = \emptyset,$

and

2.4.5 $$\{u_1^{-1}, u_2^{-1}\} \neq \{u_3, u_4\}.$$

By 1.4.6, every commutator in $F'' \setminus [F'',F]$ can be written in the form 2.4.2.

If $C = [v_1^{-1}, v_2^{-1}; v_3, v_4, \ldots, v_r]$ $(r \geq 4, v_i \in XUX^{-1})$ then, by using 1.4.4 and/or 1.4.3 and 1.4.7, C can be written as a product of commutators satisfying 2.4.3 (as in 2.3.1). Further, since by 1.4.6

$$\begin{aligned} &[v_1^{-1}, v_2^{-1}; v_3, v_4, \ldots, v_r] \\ = \ &[v_1^{-1}, v_2^{-1}, v_r^{-1}, \ldots, v_5^{-1}; v_3, v_4], \end{aligned}$$

the same argument shows that C can be written as a product of commutators satisfying 2.4.4.

Finally, if in C

$$\{v_1^{-1}, v_2^{-1}\} = \{v_3, v_4\},$$

then

$$C = [v_1^{-1},v_2^{-1};v_1^{-1},v_2^{-1},v_5,\ldots,v_r]^{\delta} \quad (\delta \in \{1,-1\}) \quad \text{by } 1.4.3$$

$$= [v_1^{-1},v_2^{-1};v_1^{-1},v_5,v_2^{-1},\ldots,v_r]^{\delta}$$

$$[v_1^{-1},v_2^{-1};v_5,v_2^{-1},v_1^{-1},\ldots,v_r]^{\delta} \quad \text{by } 1.4.5$$

$$= [v_1^{-1},v_2^{-1},v_2;v_1^{-1},v_5,\ldots,v_r]^{\delta}$$

$$[v_2^{-1},v_1^{-1},v_1;v_2^{-1},v_5,\ldots,v_r]^{\delta} \quad \text{by } 1.4.4,\ 1.4.6 \text{ and } 1.4.3$$

$$= [v_1^{-1},v_2^{-1};v_1^{-1},v_5,\ldots,v_r]^{-\delta}$$

$$[v_1^{-1},v_2;v_1^{-1},v_5,\ldots,v_r]^{-\delta}$$

$$[v_2^{-1},v_1^{-1};v_2^{-1},v_5,\ldots,v_r]^{-\delta}$$

$$[v_2^{-1},v_1;v_2^{-1},v_5,\ldots,v_r]^{-\delta} \quad \text{by } 1.4.7,$$

which is a power product of commutators satisfying 2.4.3, 2.4.4 and 2.4.5.

If a commutator C of the form 2.4.2 satisfies 2.4.3, 2.4.4 and 2.4.5 then I call it a "special" commutator.

If $C = [v_1^{-1}, v_2^{-1}; v_3, v_4, \ldots, v_s]$ $(s \geq 4,\ v_i \in XUX^{-1})$ is a special commutator then I say that $C \in$ Category I, if

(1) $v_i \neq v_j$ for $i \neq j$

(2) $\left\{v_1, \ldots, v_s\right\} \cap \left\{v_1^{-1}, \ldots, v_s^{-1}\right\} = \emptyset$;

$C \in$ Category II, if

(1) $v_1, \ldots, v_s$ are not all distinct,

(2) $\left\{v_1^{-1}, v_2^{-1}\right\} \cap \left\{v_3, v_4\right\} = \emptyset$;

$C \in$ Category III, if

(1) $\left\{v_1^{-1}, v_2^{-1}\right\} \cap \left\{v_3, v_4\right\} = v_1^{-1}$ or v_2^{-1},

(2) $v_i \neq v_j$ for $i \neq j$;

and

$C \in$ Category IV, if

(1) $\left\{v_1^{-1}, v_2^{-1}\right\} \cap \left\{v_3, v_4\right\} = v_1^{-1}$ or v_2^{-1}

(2) $v_i = v_j$ for some $i \neq j$;

clearly, for each factor $C_i^{\delta_i}$ of w, C_i belongs to one of the above categories only.

Next I write w as

2.4.6 $\qquad w = w_1 w_2 \hat{w}$

where w_1 is a power product of special commutators all of maximum weight (say) r^*, w_2 is a power product of special commutators all of weight strictly less than r^* and $\hat{w} \in [F'',F]$. Now my aim is to prove that there is a representation of w in which w_1 is empty; this then implies that $w = \hat{w} \in [F'',F]$. I suppose that w_1 is non-empty and write

$$w_1 = w_{11}w_{12}w_{13}w_{14}$$

where w_{11}, w_{12}, w_{13} and w_{14} are respectively power products of commutators in Cat. I, Cat. II, Cat. III and Cat. IV.

I shall prove in four different steps that each of w_{11}, w_{12}, w_{13} and w_{14} is empty thus contradicting the choice of w_1. In each of the steps I, II, III and IV I assume respectively w_{11}, w_{12}, w_{13} and w_{14} to be non-empty and obtain a contradiction in each case.

First I prove the following results:

2.4.7 If $C = [v_1^{-1}, v_2^{-1}; v_3, v_4, \ldots, v_s]$ $(s \geq 5,\ v_i \in XUX^{-1})$ is a special commutator then for any pair v_i, v_j for $i \neq j$, C can be written as a power product of special commutators of the form

$$[v_1'^{-1}, v_2'^{-1}; v_3', v_4', \ldots, v_{s_1}']$$

where $s_1 \leq s$, $v_1', \ldots, v_{s_1}' \in \{v_1, \ldots, v_s\}$ and either (1) $v_i \in \{v_1', v_2'\}$ and $v_j \in \{v_3', v_4'\}$

or (2) $v_j \in \{v_1', v_2'\}$ and $v_i \in \{v_3', v_4'\}$.

<u>Proof</u>. (i) if $v_i \in \{v_1, v_2\}$ and $v_j \in \{v_5, \ldots, v_s\}$ then

$$C = [v_1^{-1}, v_2^{-1}; v_3, v_4, \ldots, v_j, \ldots, v_s]$$

$$= [v_1^{-1}, v_2^{-1}; v_3, v_j, v_4, \ldots, v_s] \cdot$$

$$\cdot [v_1^{-1}, v_2^{-1}; v_j, v_4, v_3, \ldots, v_s] \qquad \text{by } 1.4.4,\ 1.4.5;$$

(ii) if $v_i \in \{v_3, v_4\}$ and $v_j \in \{v_5, \ldots, v_s\}$ then

$$C = [v_1^{-1}, v_2^{-1}; v_3, v_4, \ldots, v_j, \ldots, v_s]$$

$$= [v_1^{-1}, v_j^{-1}; v_3, v_4, \ldots, v_2, \ldots, v_s] .$$

$$\cdot [v_j^{-1}, v_2^{-1}; v_3, v_4, \ldots, v_1, \ldots, v_s] \qquad \text{by } 1.4.4,\ 1.4.6 \text{ and } 1.4.5;$$

(iii) if $v_i, v_j \in \{v_5, \ldots, v_s\}$ then

$$C = [v_1^{-1}, v_2^{-1}; v_3, v_4, \ldots, v_i, \ldots, v_j, \ldots, v_s] \quad (s \geq 6)$$

$$= [v_1^{-1}, v_2^{-1}, v_j^{-1}; v_3, v_4, v_i, \ldots, v_s] \qquad \text{by } 1.4.4,\ 1.4.6$$

$$= [v_1^{-1},v_j^{-1};v_3,v_i,\ldots,v_4,\ldots,v_2,\ldots,v_s]$$

$$[v_1^{-1},v_j^{-1};v_i,v_4,\ldots,v_3,\ldots,v_2,\ldots,v_s]$$

$$[v_j^{-1},v_2^{-1};v_3,v_i,\ldots,v_4,\ldots,v_1,\ldots,v_s]$$

$$[v_j^{-1},v_2^{-1};v_i,v_4,\ldots,v_3,\ldots,v_1,\ldots,v_s]$$

by 1.4.5, 1.4.6 and 1.4.4;

(iv) if $\{v_i,v_j\} = \{v_1,v_2\}$

then

$$C = [v_i^{-1},v_j^{-1};v_3,v_4,\ldots,v_s]^{\delta} \quad (\delta \in \{1,-1\}, \quad s \geq 5)$$

by 1.4.3

$$= [v_i^{-1},v_j^{-1},v_s^{-1};v_3,v_4,\ldots,v_{s-1}]^{\delta} \quad \text{by } 1.4.6$$

$$= [v_i^{-1},v_s^{-1};v_3,v_4,v_j,\ldots,v_{s-1}]^{\delta}$$

$$[v_s^{-1},v_j^{-1};v_3,v_4,v_i,\ldots,v_{s-1}]^{\delta} \quad \text{by } 1.4.5,\ 1.4.6\ 1.4.4$$

$$= [v_i^{-1},v_s^{-1};v_3,v_j,\ldots,v_{s-1},v_4]^{\delta}$$

$$[v_i^{-1},v_s^{-1};v_j,v_4,\ldots,v_{s-1},v_3]^{\delta}$$

$$[v_s^{-1},v_j^{-1};v_3,v_i,\ldots,v_{s-1},v_4]^{\delta}$$

$$[v_s^{-1},v_j^{-1};v_i,v_4,\ldots,v_{s-1},v_3]^{\delta} \quad \text{by } 1.4.5,\ 1.4.4;$$

(v) if $\{v_i, v_j\} = \{v_3, v_4\}$

then

$$\begin{aligned} C &= [v_1^{-1}, v_2^{-1}; v_i, v_j, \ldots, v_s]^{\delta} && \text{by } 1.4.3 \\ &= [v_1^{-1}, v_2^{-1}, v_s^{-1}, \ldots, v_5^{-1}; v_i, v_j]^{\delta} && \text{by } 1.4.6 \\ &= [v_i, v_j; v_1^{-1}, v_2^{-1}, v_s^{-1}, \ldots, v_5^{-1}]^{-\delta} && \text{by } 1.4.3 \end{aligned}$$

and the rest of the proof is similar to that of (iv); (vi) if C is in Cat. III or Cat. IV, then in the cases (i) to (v) above some of the commutator factors will no longer remain special; but by 1.4.4, 1.4.6, 1.4.3 and 1.4.7 these can again be written as a product of special commutators of weight one less than the weight of C and in each factor of smaller weight the conditions (1) and (2) of 2.4.7 are not affected.

2.4.8 If w_1', w_2' are in F, then

(i) $$\alpha([w_1', w_2']) = \underset{\sim}{w}_1'^{-1}(-1 + \underset{\sim}{w}_2'^{-1})\alpha(w_1') - \underset{\sim}{w}_2'^{-1}(-1 + \underset{\sim}{w}_1'^{-1})\alpha(w_2')$$

(ii) $$\alpha([v_1, v_2, \ldots, v_s]) = (-1 + \underset{\sim}{v}_3^{-1}) \ldots (-1 + \underset{\sim}{v}_s^{-1})\alpha([v_1, v_2])$$

for $s \geq 2$ and $v_i \in XUX^{-1}$.

<u>Proof</u>. The proof of (i) is similar to that of 2.3.3 and the proof of (ii) is similar to that of 2.3.4(i).

2.4.9 If $C^{\delta} = [v_1^{-1}, v_2^{-1}; v_3, v_4, \ldots, v_s]^{\delta}$ $(s \geq 4,\ v_i \in XUX^{-1},$

$\delta \in \{1,-1\})$

then

(i) $$\gamma(C^{\delta}) = \delta(-1 + \underset{\sim}{v}_5^{-1}) \ldots (-1 + \underset{\sim}{v}_s^{-1})\beta([v_1^{-1}, v_2^{-1}])\alpha([v_3, v_4])$$

$$-\delta(-1 + \underset{\sim}{v}_5) \ldots (-1 + \underset{\sim}{v}_s)\beta([v_3, v_4])\alpha([v_1^{-1}, v_2^{-1}]),$$

(ii) if $C^{\delta} = [x_i^{-\varepsilon_i}, x_j^{-\varepsilon_j}; x_k^{\varepsilon_k}, x_\ell^{\varepsilon_\ell}, v_5, \ldots, v_s]^{\delta}$ $(\varepsilon_i, \varepsilon_j, \varepsilon_k, \varepsilon_\ell \in \{1,-1\})$

and if i, j, k, ℓ are all distinct then

$$\gamma(C^{\delta}) = x_{ik}(C^{\delta})\lambda_i\mu_k + x_{i\ell}(C^{\delta})\lambda_i\mu_\ell$$

$$+ x_{jk}(C^{\delta})\lambda_j\mu_k + x_{j\ell}(C^{\delta})\lambda_j\mu_\ell$$

$$+ x_{ki}(C^{\delta})\lambda_k\mu_i + x_{kj}(C^{\delta})\lambda_k\mu_j$$

$$+ x_{\ell i}(C^{\delta})\lambda_\ell\mu_i + x_{\ell j}(C^{\delta})\lambda_\ell\mu_j ,$$

where

$$x_{\ell i}(C^{\delta}) = -\delta\varepsilon_\ell\varepsilon_i \underset{\sim}{x}_i^{\varepsilon_i - 1/2} \underset{\sim}{x}_\ell^{\varepsilon_\ell - 1/2}(-1 + \underset{\sim}{x}_j^{\varepsilon_j})(-1 + \underset{\sim}{x}_k^{\varepsilon_k}).$$

$$.(-1 + \underset{\sim}{v}_5) \ldots (-1 + \underset{\sim}{v}_s);$$

$$x_{\ell j}(C^{\delta}) = \delta\varepsilon_\ell\varepsilon_j \underset{\sim}{x}_j^{\varepsilon_j - 1/2} \underset{\sim}{x}_\ell^{\varepsilon_\ell - 1/2}(-1 + \underset{\sim}{x}_i^{\varepsilon_i})(-1 + \underset{\sim}{x}_k^{\varepsilon_k}).$$

$$.(-1 + \underset{\sim}{v}_5) \ldots (-1 + \underset{\sim}{v}_s);$$

and

$$\varkappa_{kj}(C^{\delta}) = -\delta\varepsilon_k\varepsilon_j \underset{\sim}{x}_j^{\varepsilon_j - 1/2} \underset{\sim}{x}_k^{\varepsilon_k - 1/2}(-1 + \underset{\sim}{x}_i^{\varepsilon_i})(-1 + \underset{\sim}{x}_\ell^{\varepsilon_\ell}).$$

$$.(-1 + \underset{\sim}{v}_5) \ \ldots \ (-1 + \underset{\sim}{v}_s)$$

(the other coefficients have similar forms but are not needed).

Proof.

(i) $$\gamma(C^{\delta}) = \delta\beta([v_1^{-1}, v_2^{-1}])\alpha([v_3, v_4, \ldots, v_s])$$

$$-\delta\beta([v_3, v_4, \ldots, v_s])\alpha([v_1^{-1}, v_2^{-1}])$$

by 2.1.9(iii), 2.2.1, 2.2.2 and 2.2.3

$$= \delta(-1 + \underset{\sim}{v}_5^{-1}) \ \ldots \ (-1 + \underset{\sim}{v}_s^{-1})\beta([v_1^{-1}, v_2^{-1}])\alpha([v_3, v_4])$$

$$-\delta(-1 + \underset{\sim}{v}_5) \ \ldots \ (-1 + \underset{\sim}{v}_s)\beta([v_3, v_4])\alpha([v_1^{-1}, v_2^{-1}])$$

by 2.4.8(ii), 2.3.4;

(ii) if $C^{\delta} = [x_i^{-\varepsilon_i}, x_j^{-\varepsilon_j}; x_k^{\varepsilon_k}, x_\ell^{\varepsilon_\ell}, v_5, \ldots, v_s]^{\delta}$

then

$$\gamma(C^{\delta}) = \delta(-1 + \underset{\sim}{v}_5^{-1}) \ \ldots \ (-1 + \underset{\sim}{v}_s^{-1})\beta([x_i^{-\varepsilon_i}, x_j^{-\varepsilon_j}])\alpha([x_k^{\varepsilon_k}, x_\ell^{\varepsilon_\ell}])$$

$$-\delta(-1 + \underset{\sim}{v}_5) \ \ldots \ (-1 + \underset{\sim}{v}_s)\beta([x_k^{\varepsilon_k}, x_\ell^{\varepsilon_\ell}])\alpha([x_i^{-\varepsilon_i}, x_j^{-\varepsilon_j}])$$

by 2.4.9(i)

$$= \delta(-1 + \underset{\sim}{v}_5^{-1}) \ldots (-1 + \underset{\sim}{v}_s^{-1})\Big(\big((-1 + \underset{\sim}{x}_j^{-\varepsilon_j})\beta(x_i^{-\varepsilon_i}) - (-1 + \underset{\sim}{x}_i^{-\varepsilon_i})\beta(x_j^{-\varepsilon_j})\big).$$

$$.\big(\underset{\sim}{x}_k^{-\varepsilon_k}(-1 + \underset{\sim}{x}_\ell^{-\varepsilon_\ell})\alpha(x_k^{\varepsilon_k}) - \underset{\sim}{x}_\ell^{-\varepsilon_\ell}(-1 + \underset{\sim}{x}_k^{-\varepsilon_k})\alpha(x_\ell^{\varepsilon_\ell})\big)\Big)$$

$$-\delta(-1 + \underset{\sim}{v}_5) \ldots (-1 + \underset{\sim}{v}_s)\Big(\big((-1 + \underset{\sim}{x}_\ell^{\varepsilon_\ell})\beta(x_k^{\varepsilon_k}) - (-1 + \underset{\sim}{x}_k^{\varepsilon_k})\beta(x_\ell^{\varepsilon_\ell})\big).$$

$$.\big(\underset{\sim}{x}_i^{\varepsilon_i}(-1 + \underset{\sim}{x}_j^{\varepsilon_j})\alpha(x_i^{-\varepsilon_i}) - \underset{\sim}{x}_j^{\varepsilon_j}(-1 + \underset{\sim}{x}_i^{\varepsilon_i})\alpha(x_j^{-\varepsilon_j})\big)\Big)$$

by 2.4.8(i), 2.3.3

$$= \delta(-1 + \underset{\sim}{v}_5^{-1}) \ldots (-1 + \underset{\sim}{v}_s^{-1})\Big(\big(-\varepsilon_i\underset{\sim}{x}_i^{-\varepsilon_i - 1/2}(-1 + \underset{\sim}{x}_j^{-\varepsilon_j})\lambda_i + \varepsilon_j\underset{\sim}{x}_j^{-\varepsilon_j - 1/2}.$$

$$\cdot(-1 + \underset{\sim}{x}_i^{-\varepsilon_i})\lambda_j\big).$$

$$.\big(\varepsilon_k\underset{\sim}{x}_k^{-\varepsilon_k - 1/2}(-1 + \underset{\sim}{x}_\ell^{-\varepsilon_\ell})\mu_k - \varepsilon_\ell\underset{\sim}{x}_\ell^{-\varepsilon_\ell - 1/2}(-1 + \underset{\sim}{x}_k^{-\varepsilon_k})\mu_\ell\big)\Big)$$

$$-\delta(-1 + \underset{\sim}{v}_5) \ldots (-1 + \underset{\sim}{v}_s)\Big(\big(\varepsilon_k\underset{\sim}{x}_k^{\varepsilon_k - 1/2}(-1 + \underset{\sim}{x}_\ell^{\varepsilon_\ell})\lambda_k - \varepsilon_\ell\underset{\sim}{x}_\ell^{\varepsilon_\ell - 1/2}\cdot$$

$$\cdot(-1 + \underset{\sim}{x}_k^{\varepsilon_k})\lambda_\ell\big).$$

$$.\big(-\varepsilon_i\underset{\sim}{x}_i^{\varepsilon_i - 1/2}(-1 + \underset{\sim}{x}_j^{\varepsilon_j})\mu_i + \varepsilon_j\underset{\sim}{x}_j^{\varepsilon_j - 1/2}(-1 + \underset{\sim}{x}_i^{\varepsilon_i})\mu_j\big)\Big)$$

by 2.1.9(i), (ii) and 2.1.7(ii),

thus the result follows.

$\bigcirc$

As a consequence to the proof of 2.4.9(ii) I get

2.4.10 (i) if $C^\delta = [x_i^{-\varepsilon_i}, x_j^{-\varepsilon_j}; x_i^{\varepsilon_i'}, x_\ell^{\varepsilon_\ell}, v_5, \ldots, v_s]^\delta (\varepsilon_i, \varepsilon_j, \varepsilon_i', \varepsilon_\ell \in \{1,-1\})$

and if i,j,ℓ are all distinct, then

$$x_{ii}(C^\delta) = \delta\varepsilon_i\varepsilon_i' \underset{\sim}{x}_i^{\varepsilon_i'-1/2} \underset{\sim}{x}_i^{\varepsilon_i-1/2} (-1 + \underset{\sim}{x}_j^{\varepsilon_j})(-1 + \underset{\sim}{x}_\ell^{\varepsilon_\ell}).$$
$$.(-1 + \underset{\sim}{v}_5) \ldots (-1 + \underset{\sim}{v}_s)$$
$$-\delta\varepsilon_i\varepsilon_i' \underset{\sim}{x}_i^{-\varepsilon_i-1/2} \underset{\sim}{x}_i^{-\varepsilon_i'-1/2} (-1 + \underset{\sim}{x}_j^{-\varepsilon_j})(-1 + \underset{\sim}{x}_\ell^{-\varepsilon_\ell}).$$
$$.(-1 + \underset{\sim}{v}_5^{-1}) \ldots (-1 + \underset{\sim}{v}_s^{-1});$$

$$x_{ij}(C^\delta) = -\delta\varepsilon_i'\varepsilon_j \underset{\sim}{x}_i^{\varepsilon_i'-1/2} \underset{\sim}{x}_j^{\varepsilon_j-1/2} (-1 + \underset{\sim}{x}_i^{\varepsilon_i})(-1 + \underset{\sim}{x}_\ell^{\varepsilon_\ell}).$$
$$.(-1 + \underset{\sim}{v}_5) \ldots (-1 + \underset{\sim}{v}_s);$$

$$x_{\ell i}(C^\delta) = -\delta\varepsilon_\ell\varepsilon_i \underset{\sim}{x}_\ell^{\varepsilon_\ell-1/2} \underset{\sim}{x}_i^{\varepsilon_i-1/2} (-1 + \underset{\sim}{x}_i^{\varepsilon_i'})(-1 + \underset{\sim}{x}_j^{\varepsilon_j}).$$
$$.(-1 + \underset{\sim}{v}_5) \ldots (-1 + \underset{\sim}{v}_s); \quad \text{and}$$

$$x_{\ell j}(C^\delta) = \delta\varepsilon_\ell\varepsilon_j \underset{\sim}{x}_j^{\varepsilon_j-1/2} \underset{\sim}{x}_\ell^{\varepsilon_\ell-1/2} (-1 + \underset{\sim}{x}_i^{\varepsilon_i})(-1 + \underset{\sim}{x}_i^{\varepsilon_i'}).$$
$$.(-1 + \underset{\sim}{v}_5) \ldots (-1 + \underset{\sim}{v}_s);$$

(ii) if $C^\delta = [x_i^{-\varepsilon_i}, x_j^{-\varepsilon_j}; x_i^{\varepsilon'_i}, x_j^{\varepsilon'_j}, v_5, \ldots, v_s]^\delta$

$(\varepsilon_i, \varepsilon_j, \varepsilon'_i, \varepsilon'_j \in \{1,-1\}$ and $i \neq j)$ then

$$x_{ii}(C^\delta) = \delta\varepsilon_i\varepsilon'_i \underset{\sim}{x}_i^{\varepsilon_i - 1/2} \underset{\sim}{x}_i^{\varepsilon'_i - 1/2}(-1 + \underset{\sim}{x}_j^{\varepsilon'_j})(-1 + \underset{\sim}{x}_j^{\varepsilon_j}).$$

$$.(-1 + \underset{\sim}{v}_5) \ldots (-1 + \underset{\sim}{v}_s)$$

$$-\delta\varepsilon_i\varepsilon'_i \underset{\sim}{x}_i^{-\varepsilon_i - 1/2} \underset{\sim}{x}_i^{-\varepsilon'_i - 1/2}(-1 + \underset{\sim}{x}_j^{-\varepsilon_j})(-1 + \underset{\sim}{x}_j^{-\varepsilon'_j}).$$

$$.(-1 + \underset{\sim}{v}_5^{-1}) \ldots (-1 + \underset{\sim}{v}_s);$$

$$x_{ij}(C^\delta) = -\delta\varepsilon'_i\varepsilon_j \underset{\sim}{x}_i^{\varepsilon'_i - 1/2} \underset{\sim}{x}_j^{\varepsilon_j - 1/2}(-1 + \underset{\sim}{x}_j^{\varepsilon'_j})(-1 + \underset{\sim}{x}_i^{\varepsilon_i}).$$

$$.(-1 + \underset{\sim}{v}_5) \ldots (-1 + \underset{\sim}{v}_s)$$

$$+ \delta\varepsilon_i\varepsilon'_j \underset{\sim}{x}_i^{-\varepsilon_i - 1/2} \underset{\sim}{x}_j^{-\varepsilon'_j - 1/2}(-1 + \underset{\sim}{x}_j^{-\varepsilon_j})(-1 + \underset{\sim}{x}_i^{-\varepsilon'_i}).$$

$$.(-1 + \underset{\sim}{v}_5^{-1}) \ldots (-1 + \underset{\sim}{v}_s); \quad \text{and}$$

$$x_{jj}(C^\delta) = \delta\varepsilon'_j\varepsilon_j \underset{\sim}{x}_j^{\varepsilon'_j - 1/2} \underset{\sim}{x}_j^{\varepsilon_j - 1/2}(-1 + \underset{\sim}{x}_i^{\varepsilon'_i})(-1 + \underset{\sim}{x}_i^{\varepsilon_i}).$$

$$.(-1 + \underset{\sim}{v}_5) \ldots (-1 + \underset{\sim}{v}_s)$$

$$-\delta\varepsilon_j\varepsilon'_j \underset{\sim}{x}_j^{-\varepsilon_j - 1/2} \underset{\sim}{x}_j^{-\varepsilon'_j - 1/2}(-1 + \underset{\sim}{x}_i^{-\varepsilon_i})(-1 + \underset{\sim}{x}_i^{-\varepsilon'_i}).$$

$$.(-1 + \underset{\sim}{v}_5^{-1}) \ldots (-1 + \underset{\sim}{v}_s)$$

(the other coefficients in (i), (ii) have similar forms but are not needed).

DETAILS OF STEP I.

First of all I prove

2.4.11 If $C^{\delta} = [u_1^{-1}, u_2^{-1}; u_3, u_4, \ldots, u_{t_1}]^{\delta}$

$$(\delta \in \{1,-1\},\ t_1 = r^* \geq 4,\ u_i \in XUX^{-1})$$

is a factor in w_{11} then w_{11} has another factor

$$[u_{1\sigma}^{-1}, u_2^{-1}; u_3, u_{4\sigma}, \ldots, u_{t_1\sigma}]^{-\delta},$$

where σ is a permutation of $\{1,4,\ldots,t_1\}$.

Proof. If $u_2 = x_j^{\varepsilon_j}$, $u_3 = x_k^{\varepsilon_k}$ $(j \neq k)$ then

$$C^{\delta} = [u_1^{-1}, x_j^{-\varepsilon_j}; x_k^{\varepsilon_k}, u_4, \ldots, u_{t_1}]^{\delta} .$$

Since C is a special commutator of Cat. I, by 2.4.9

$$\chi_{kj}(C^{\delta}) = -\delta\varepsilon_j\varepsilon_k \underset{\sim}{x}_j^{\varepsilon_j - 1/2} \underset{\sim}{x}_k^{\varepsilon_k - 1/2}(-1 + \underset{\sim}{u}_1)(-1 + \underset{\sim}{u}_4) \ldots (-1 + \underset{\sim}{u}_{t_1});$$

and a term of $\chi_{kj}(C^{\delta})$ is

$$-\delta\varepsilon_j\varepsilon_k \underset{\sim}{x}_j^{\varepsilon_j - 1/2} \underset{\sim}{x}_k^{\varepsilon_k - 1/2} \underset{\sim}{u}_1 \underset{\sim}{u}_4 \ldots \underset{\sim}{u}_{t_1},$$

which is the only term of maximum length precisely $t_1 - 2 + \tau$, where

$$\tau = \begin{cases} 0 & \text{if } \varepsilon_j = \varepsilon_k = 1 \\ 1 & \text{if } \varepsilon_j + \varepsilon_k = 0 \\ 2 & \text{if } \varepsilon_j = \varepsilon_k = -1 . \end{cases}$$

Since, by 2.1.8(iii), 2.2.1 and 2.2.2

$$\gamma(w) = \gamma(C_1^{\delta_1}) + \ldots + \gamma(C_m^{\delta_m})$$

and by hypothesis $\gamma(w) = 0$, then it follows (by 2.1.14(iii)) that

$$0 = \chi_{kj}(w) = \chi_{kj}(C_1^{\delta_1}) + \ldots + \chi_{kj}(C_m^{\delta_m}).$$

Thus, there is a factor (say) $C_{kj}^{\delta_{kj}}$ in w which is different from C^{δ} such that a term of $\chi_{kj}(C_{kj}^{\delta_{kj}})$ is

$$+\delta\varepsilon_j\varepsilon_k \underset{\sim}{x}_j^{\varepsilon_j - 1/2} \underset{\sim}{x}_k^{\varepsilon_k - 1/2} \underset{\sim}{u}_1 \underset{\sim}{u}_4 \cdots \underset{\sim}{u}_{t_1} .$$

By 2.4.9, 1.4.3, 1.2.1 and 1.4.6 I can assume that $C_{kj}^{\delta_{kj}}$ has the form

$$C_{kj}^{\delta_{kj}} = [v_1^{-1}, x_j^{-\varepsilon'_j}; x_k^{\varepsilon'_k}, v_2, \ldots, v_s]^{\delta_{kj}} \quad (s \geq 2)$$

where $\varepsilon'_j, \varepsilon'_k, \delta_{kj} \in \{1,-1\}$, $v_1, \ldots, v_s \in XUX^{-1}$ and by 2.4.1, $s + 2 \leq t_1$.

If $v_1 = x_k^{\varepsilon''_k}$ and $v_2 = x_j^{\varepsilon''_j}$, then

$$\{u_1, u_4, \ldots, u_{t_1}\} \subseteq \{v_3, \ldots, v_s\}$$

(because $\{u_1, u_4, \ldots, u_{t_1}\} \cap \{x_j, x_j^{-1}, x_k, x_k^{-1}\} = \emptyset$), so

$$s - 2 \geq t_1 - 2; \quad s + 2 \geq t_1 + 2 \quad \text{which is not possible}$$

since $s + 2 \leq t_1$. It follows, therefore, that

$$\{v_1, v_2\} \neq \{x_k^{\varepsilon''_k}, x_j^{\varepsilon''_j}\}.$$

By 2.4.9 (or 2.4.10(i))

$$\chi_{kj}(C_{kj}^{\delta_{kj}}) = -\delta_{kj}\varepsilon'_j\varepsilon'_k \underset{\sim}{x}_j^{\varepsilon'_j - 1/2} \underset{\sim}{x}_k^{\varepsilon'_k - 1/2}(-1 + \underset{\sim}{v}_1) \ldots (-1 + \underset{\sim}{v}_s);$$

and by hypothesis, a term of $\chi_{kj}(C_{kj}^{\delta_{kj}})$ is

$$+\delta\varepsilon_j\varepsilon_k \underset{\sim}{x}_j^{\varepsilon_j - 1/2} \underset{\sim}{x}_k^{\varepsilon_k - 1/2} \underset{\sim}{u}_1 \underset{\sim}{u}_4 \ldots \underset{\sim}{u}_{t_1},$$

thus it follows that

$$\{u_1, u_4, \ldots, u_{t_1}\} \subseteq \{v_1, \ldots, v_s\}$$

which gives

$$s \geq t_1 - 2; \quad s + 2 \geq t_1; \quad s + 2 = t_1 \quad (\text{since } s + 2 \leq t_1);$$

$$\{u_1, u_4, \ldots, u_{t_1}\} = \{v_1, \ldots, v_s\} .$$

This now gives

$$\varepsilon_j' = \varepsilon_j, \quad \varepsilon_k' = \varepsilon_k \quad \text{and hence also} \quad \delta_{kj} = -\delta.$$

Thus

$$C_{kj}^{\delta_{kj}} = [u_{1\sigma}^{-1}, x_j^{-\varepsilon_j}; x_k^{\varepsilon_k}, u_{4\sigma}, \ldots, u_{t_1\sigma}]^{-\delta}$$

$$= [u_{1\sigma}^{-1}, u_2^{-1}; u_3, u_{4\sigma}, \ldots, u_{t_1\sigma}]^{-\delta}$$

as was required.

Now, since by 1.4.3

$$[u_1^{-1}, u_2^{-1}; u_3, u_4, \ldots, u_{t_1}]^{\delta}$$
$$= [u_1^{-1}, u_2^{-1}; u_4, u_3, \ldots, u_{t_1}]^{-\delta}$$
$$= [u_2^{-1}, u_1^{-1}; u_4, u_3, \ldots, u_{t_1}]^{\delta}$$
$$= [u_2^{-1}, u_1^{-1}; u_3, u_4, \ldots, u_{t_1}]^{-\delta};$$

I get as a consequence of 2.4.11 the following result

2.4.12 If $C^{\delta} = [u_1^{-1}, u_2^{-1}; u_3, u_4, \ldots, u_{t_1}]^{\delta}$ $(t_1 = r^* \geq 4)$ is a factor of w_{11} then w_{11} also has the following four factors

$$[u_{1\sigma_1}^{-1}, u_2^{-1}; u_3, u_{4\sigma_1}, \ldots, u_{t_1\sigma_1}]^{-\delta}$$

where σ_1 is a permutation of $\{1, 4, \ldots, t_1\}$;

$$[u^{-1}_{1\sigma_2}, u^{-1}_2; u_{3\sigma}, u_4, u_{5\sigma_2}, \ldots, u_{t_1\sigma_2}]^{-\delta}$$

where σ_2 is a permutation of $\{1,3,5,\ldots,t_1\}$;

$$[u^{-1}_1, u^{-1}_{2\sigma_3}; u_3, u_{4\sigma_3}, \ldots, u_{t_1\sigma_3}]^{-\delta}$$

where σ_3 is a permutation of $\{2,4,\ldots,t_1\}$ and

$$[u^{-1}_1, u^{-1}_{2\sigma_4}; u_{3\sigma_4}, u_4, u_{5\sigma_4}, \ldots, u_{t_1\sigma_4}]^{-\delta}$$

where σ_4 is a permutation of $\{2,3,5,\ldots,t_1\}$.

Now, I shall complete the details of Step I. Suppose w_{11} is non-empty, then w_{11} is a power product of special commutators each of weight $t_1 (= r^*)$. There are two cases to be considered

Case 1. When $t_1 = 4$

and Case 2. When $t_1 \geq 5$.

<u>CASE 1</u>. By hypothesis, there is a factor (say) $C_1^{\delta_1}$ in w_{11} given by

$$C_1^{\delta_1} = [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_3}^{\varepsilon_3}, x_{i_4}^{\varepsilon_4}]^{\delta_1} \quad (i_j \neq i_k \text{ for } j \neq k).$$

Using 2.4.12, I get in w_{11}

$$[x_{i_1}^{-\varepsilon_1}, x_{i_4}^{-\varepsilon_4}; x_{i_3}^{\varepsilon_3}, x_{i_2}^{\varepsilon_2}]^{-\delta_1} \quad \text{(by considering } x_{i_3 i_1}(C_1^{\delta_1}));$$

$$[x_{i_1}^{-\varepsilon_1}, x_{i_3}^{-\varepsilon_3}; x_{i_2}^{\varepsilon_2}, x_{i_4}^{\varepsilon_4}]^{-\delta} \quad \text{(by considering } x_{i_4 i_1}(C_1^{\delta_1}));$$

$$[x_{i_4}^{-\varepsilon_4}, x_{i_2}^{-\varepsilon_2}; x_{i_3}^{\varepsilon_3}, x_{i_1}^{\varepsilon_1}]^{-\delta_1} \quad \text{(by considering } x_{i_3 i_2}(C_1^{\delta_1}));$$

$$[x_{i_3}^{-\varepsilon_3}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{\varepsilon_1}, x_{i_4}^{\varepsilon_4}]^{-\delta_1} \quad \text{(by considering } x_{i_4 i_2}(C_1^{\delta_1}));$$

and

$$[x_{i_3}^{-\varepsilon_3}, x_{i_4}^{-\varepsilon_4}; x_{i_1}^{\varepsilon_1}, x_{i_2}^{\varepsilon_2}]^{\delta_1}$$

$$\text{(by considering } x_{i_1 i_3}([x_{i_3}^{-\varepsilon_3}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{\varepsilon_1}, x_{i_4}^{\varepsilon_4}]^{-\delta_1})).$$

Now

$$[x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_3}^{\varepsilon_3}, x_{i_4}^{\varepsilon_4}]^{\delta_1} [x_{i_1}^{-\varepsilon_1}, x_{i_4}^{-\varepsilon_4}; x_{i_3}^{\varepsilon_3}, x_{i_2}^{\varepsilon_2}]^{-\delta_1} .$$

$$.[x_{i_1}^{-\varepsilon_1}, x_{i_3}^{-\varepsilon_3}; x_{i_2}^{\varepsilon_2}, x_{i_4}^{\varepsilon_4}]^{-\delta_1} . [x_{i_4}^{-\varepsilon_4}, x_{i_2}^{-\varepsilon_2}; x_{i_3}^{\varepsilon_3}, x_{i_1}^{\varepsilon_1}]^{-\delta_1} .$$

$$.[x_{i_3}^{-\varepsilon_3}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{\varepsilon_1}, x_{i_4}^{\varepsilon_4}]^{-\delta_1} . [x_{i_3}^{-\varepsilon_3}, x_{i_4}^{-\varepsilon_4}; x_{i_1}^{\varepsilon_1}, x_{i_2}^{\varepsilon_2}]^{\delta_1} .$$

$$= [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_3}^{\varepsilon_3}, x_{i_4}^{\varepsilon_4}]^{\delta_1} . [x_{i_1}^{-\varepsilon_1}, x_{i_4}^{-\varepsilon_4}; x_{i_2}^{\varepsilon_2}, x_{i_3}^{\varepsilon_3}]^{\delta_1} .$$

$$.[x_{i_1}^{-\varepsilon_1},x_{i_3}^{-\varepsilon_3};x_{i_4}^{\varepsilon_4},x_{i_2}^{\varepsilon_2}]^{\delta_1}.[x_{i_4}^{-\varepsilon_4},x_{i_2}^{-\varepsilon_2};x_{i_1}^{\varepsilon_1},x_{i_3}^{\varepsilon_3}]^{\delta_1}.$$

$$.[x_{i_3}^{-\varepsilon_3},x_{i_2}^{-\varepsilon_2};x_{i_4}^{\varepsilon_4},x_{i_1}^{\varepsilon_1}]^{\varepsilon_1}.[x_{i_3}^{-\varepsilon_3},x_{i_4}^{-\varepsilon_4};x_{i_1}^{\varepsilon_1},x_{i_2}^{\varepsilon_2}]^{\delta_1}$$

by 1.4.3,

belongs to the verbal subgroup V by hypothesis.

<u>CASE 2</u>. By hypothesis, there is a factor (say) $C_1^{\delta_1}$ in w_{11} given by

$$C_1^{\delta_1} = [x_{i_1}^{-\varepsilon_1},x_{i_2}^{-\varepsilon_2};x_{i_3}^{\varepsilon_3},x_{i_4}^{\varepsilon_4},\ldots,x_{i_{t_1}}^{\varepsilon_{t_1}}]^{\delta_1},$$

where C_1 is a special commutator of Cat. I and $t_1 = r^* \geq 5$.

About the remaining factors of w_{11} (if any), by 2.4.7, I can make the following assertion:

2.4.13 If $[u_1^{-1},u_2^{-1};u_3,u_4,\ldots,u_{t_1}]^{\delta}$ is a factor of w_{11} such that

$$\{x_{i_1}^{\varepsilon_1'},x_{i_3}^{\varepsilon_3'}\} \subset \{u_1,\ldots,u_{t_1}\} \quad (\varepsilon_1',\varepsilon_3' \in \{1,-1\})$$

then

either (1) $x_{i_1}^{\varepsilon_1'} \in \{u_1,u_2\}$ and $x_{i_3}^{\varepsilon_3'} \in \{u_3,u_4\}$

or (2) $x_{i_3}^{\varepsilon_3'} \in \{u_1,u_2\}$ and $x_{i_1}^{\varepsilon_1'} \in \{u_3,u_4\}$.

Thus, there is a representation of w as power product of special commutators and the factors of w_{11} satisfy 2.4.13. Among such representations of w I choose one in which w_{11} consists of least number of factors and I write

$$w_{11} = C_{11}^{\delta_{11}} C_{21}^{\delta_{21}} \ldots C_{m_1 1}^{\delta_{m_1 1}} \quad (m_1 \geq 1,\ \delta_{11}, \delta_{21}, \ldots, \delta_{m_1 1} \in \{1,-1\}),$$

where $C_{11}^{\delta_{11}} = C_1^{\delta_1}$.

For simplicity of notation I write $C_1^{\delta_1}$ as

$f_1^* = [i_1^{-1}, i_2^{-1}; i_3, i_4, \ldots, i_{t_1}]^{\delta_1}$ $(i_1, \ldots, i_{t_1}$ are all distinct).

Considering $x_{i_4 i_2}(f_1^*)$ and $x_{i_4 i_1}(f_1^*)$ and using 2.4.12 and 2.4.13 respectively implies that in w_{11} there are factors f_2^* and f_3^* given by $f_2^* = [i_3^{-1}, i_2^{-1}; i_1, i_4, \ldots, i_{t_1}]^{-\delta_1}$ and

$f_3^* = [i_1^{-1}, i_{2\sigma_1}^{-1}; i_3, i_4, i_{5\sigma_1}, \ldots, i_{t_1 \sigma_1}]^{-\delta_1}$, where σ_1 is a permutation of $\{2, 5, \ldots, t_1\}$ and $2\sigma_1 \neq 2$ (for, if $2\sigma_1 = 2$ then $f_1^* f_2^* f_3^* = f_2^*$ modulo $[F'', F]$; this contradicts the choice of w_{11}).

Similarly considering $x_{i_4 i_{2\sigma_1}}(f_3^*)$ and $x_{i_3 i_{2\sigma_1}}(f_3^*)$ (and using 2.4.12, 2.4.13 respectively) implies that in w_{11} there are factors f_4^* and f_5^* given by

$$f_4^* = [i_3^{-1}, i_{2\sigma_1}^{-1}; i_1, i_4, i_{5\sigma_1}, \ldots, i_{t_1\sigma_1}]^{\delta_1}$$

and

$$f_5^* = [i_1^{-1}, i_{2\sigma_1}^{-1}; i_3, i_{4\sigma_2}, i_{5\sigma_1\sigma_2}, \ldots, i_{t_1\sigma_1\sigma_2}]^{\delta_1}$$

where σ_2 is a permutation of $\{4, 5\sigma_1, \ldots, t_1\sigma_1\}$ and $4\sigma_2 \neq 4$ (for, if $4\sigma_2 = 4$ then $f_1^* f_2^* f_3^* f_4^* f_5^* = f_1^* f_2^* f_4^*$ modulo $[F'', F]$ (by using 1.4.4), contradicting the choice of w_{11}).

Further, considering $x_{i_{4\sigma_2} i_{2\sigma_1}}(f_5^*)$ and $x_{i_{4\sigma_2} i_1}(f_5^*)$ (and using 2.4.12, 2.4.13 respectively) gives as before

$$f_6^* = [i_3^{-1}, i_{2\sigma_1}^{-1}; i_1, i_{4\sigma_2}, i_{5\sigma_1\sigma_2}, \ldots, i_{t_1\sigma_1\sigma_2}]^{-\delta_1}$$

and

$$f_7^* = [i_1^{-1}, i_{2\sigma_1\sigma_3}^{-1}; i_3, i_{4\sigma_2}, i_{5\sigma_1\sigma_2\sigma_3}, \ldots, i_{t_1\sigma_1\sigma_2\sigma_3}]^{-\delta_1}$$

where σ_3 is a permutation of $\{2\sigma_1, 5\sigma_1\sigma_1, \ldots, t_1\sigma_1\sigma_2\}$

and $2\sigma_1\sigma_3 \neq 2\sigma_1$ (for, if $2\sigma_1\sigma_3 = 2\sigma_1$ then $f_1^* f_2^* f_3^* f_4^* f_5^* f_6^* f_7^* = f_1^* f_2^* f_3^* f_4^* f_6^*$ modulo $[F'', F]$ (by 1.4.4) which contradicts the choice of w_{11}).

Finally, considering $x_{i_{4\sigma_2} i_{2\sigma_1\sigma_3}}(f_7^*)$ (and using 2.4.12, 2.4.13 respectively) gives

$$f_8^* = [i_3^{-1}, i_{2\sigma_1\sigma_3}^{-1}; i_1, i_{4\sigma_2}, i_{5_1\sigma_2\sigma_3}, \ldots, i_{t_{1\sigma_1\sigma_2\sigma_3}}]^{-\delta_1}.$$

Thus in the minimal representation of w_{11}, I have obtained a product factor

$$f_1^* \ldots f_8^* .$$

Now, I shall obtain a contradiction to the choice of w_{11} by proving that this product $f_1^* \ldots f_8^*$ can be replaced by a product with smaller number of factors of w_{11}.

By 1.4.9,

$$f_1^* f_2^* f_3^* f_4^* = [i_2^{-1}, i_{2\sigma_1}^{-1}; i_1, i_3, i_{5\sigma_1}, \ldots, i_{t_1\sigma_1}]^{\delta_1}$$

and

$$f_5^* f_6^* f_7^* f_8^* = [i_{2\sigma_1}^{-1}, i_{2\sigma_1\sigma_3}^{-1}; i_1, i_3, i_{4\sigma_2}, i_{5\sigma_1\sigma_2}, \ldots, i_{t_1\sigma_1\sigma_2}]^{\delta_1}.$$

Hence

$$f_1^* \ldots f_8^* = [i_2^{-1}, i_{2\sigma_1\sigma_3}^{-1}; i_1, i_3, i_{2\sigma_1}, i_{4\sigma_1}, i_{5\sigma_1\sigma_2}, \ldots, i_{t_1\sigma_1\sigma_2}]^{\delta_1}$$

by 1.4.10

$$=[i_2^{-1},i_j^{-1};i_1,i_3,i_4,\ldots,i_{j-1},i_{j+1},\ldots,i_t]^{\delta_1}$$

for some $j \in \{4,5,\ldots,t_1\}$,

$$= [i_1^{-1},i_2^{-1};i_3,i_4,\ldots,i_t]^{\delta_1}.$$

$$.[i_1^{-1},i_j^{-1};i_3,i_4,\ldots,i_{j-1},i_{j+1},\ldots,i_t]^{-\delta_1}.$$

$$.[i_3^{-1},i_2^{-1};i_1,i_4,\ldots,i_t]^{-\delta_1}.$$

$$.[i_3^{-1},i_j^{-1};i_1,i_4,\ldots,i_{j-1},i_{j+1},\ldots,i_t]^{\delta_1} \text{ by } 1.4.9,$$

as was required.

DETAILS OF STEP II

By the previous step I can assume that w_{11} is empty, so that

$$w = w_{12}w_{13}w_{14}w_2\hat{w}.$$

Now I suppose that w_{12} is non-empty and arrive at a contradiction.

If $C = [v_1^{-1},v_2^{-1};v_3,v_4,\ldots,v_{s_2}]$ ($s_2 \geq 5$, $v_i \in \{XUX^{-1}\}$) is a special commutator of Cat. II, then, by definition, for some $j \neq k$, $v_j = v_k$. By 2.4.7, I can write C as a power product of special commutators in Cat. II of the kind

$$[v_1'^{-1},v_2'^{-1};v_3',v_4',\ldots,v_{s_2}'] \quad (v_i' \in XUX^{-1})$$

where v_1', v_2', v_3', v_4' are not all distinct.

Thus, if $C_1^{\delta_1}$ is a factor of w_{12} then

$$C_1^{\delta_1} = [x_{i_1}^{-\varepsilon_1},x_{i_2}^{-\varepsilon_2};x_{i_1}^{\varepsilon_1},x_{i_3}^{\varepsilon_3},(r_1-1)x_{i_1}^{\varepsilon_1},r_2x_{i_2}^{\varepsilon_2},r_3x_{i_3}^{\varepsilon_3},(r_4+1)x_{i_4}^{\varepsilon_4},\ldots,(r_{t_2}+1)x_{i_{t_2}}^{\varepsilon_{t_2}}]^{\delta_1},$$

where $r_1 \geq 1$, $r_j \geq 0$ for $2 \leq j \leq t_2$, $t_2 \geq 3$, $i_j \neq i_k$ for $j \neq k$ and $\sum_{j=1}^{t_2} (r_j+1) = r^*$;

Or

$$C_1^{\delta_1} = [x_{i_1}^{-\varepsilon_1},x_{i_2}^{-\varepsilon_2};x_{i_1}^{\varepsilon_1},x_{i_2}^{\varepsilon_2},(r_1-1)x_{i_1}^{\varepsilon_1},(r_2-1)x_{i_2}^{\varepsilon_2},(r_3+1)x_{i_3}^{\varepsilon_3},\ldots,(r_{t_2}+1)x_{i_{t_2}}^{\varepsilon_{t_2}}]^{\delta_1}$$

where $r_1,r_2 \geq 1$, $r_j \geq 0$ for $3 \leq j \leq t_2$, $t_2 \geq 2$, $i_j \neq i_k$ for $j \neq k$ and $\sum_{j=1}^{t_2} (r_j+1) = r^*$.

About the remaining factors of w, again by 2.4.7, I can make the following assertion:

2.4.14. If $C^{\delta} = [v_1^{-1},v_2^{-1};v_3,v_4,\ldots,v_{s_2}]^{\delta}$ $(s_2 \geq 4,\ \delta \in \{1,-1\})$

is a factor of w such that

$$v_j = v_k = x_{i_1}^{\varepsilon_1'} \quad \text{for some} \quad j \neq k \quad (\varepsilon_1' \in \{1,-1\})$$

then

$$x_{i_1}^{\varepsilon_1'} \in \{v_1, v_2\} \quad \text{and} \quad x_{i_1}^{\varepsilon_1'} \in \{v_3, v_4\}$$

(2.4.7 is needed only when $s_2 \geq 5$, for $s_2 = 4$ the assertion 2.4.14 is obvious).

There is a representation of w as a power product of special commutators in which w_{11} is empty, w_{12} is a power product of special commutators of maximum weight in Cat. II and the factors of $w_{12}w_{13}w_{14}w_2$ satisfy 2.4.14. Of all such representations of w I choose one in which w_{12} contains least number of factors and I write

$$w_{12} = C_{12}^{\delta_{12}} C_{22}^{\delta_{22}} \ldots\ldots C_{m_2 2}^{\delta_{m_2 2}} \quad (m_2 \geq 1,\ \delta_{12}, \ldots, \delta_{m_2 2} \in \{1,-1\}),$$

where $C_{12}^{\delta_{12}} = C_1^{\delta_1}$.

I first consider the case when

$$C_1^{\delta_1} = [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{\varepsilon_1}, x_{i_3}^{\varepsilon_3}, (r_1-1)x_{i_1}^{\varepsilon_1}, r_2 x_{i_2}^{\varepsilon_2}, r_3 x_{i_3}^{\varepsilon_3}, (r_4+1)x_{i_4}^{\varepsilon_4}, \ldots, (r_{t_2}+1)x_{i_{t_2}}^{\varepsilon_{t_2}}]^{\delta_1}.$$

By 2.4.10(i)

$$x_{i_3 i_2}(C_1^{\delta_1}) = \delta_1 \varepsilon_2 \varepsilon_3 \underset{\sim}{x}_{i_2}^{\varepsilon_2 - 1/2} \underset{\sim}{x}_{i_3}^{\varepsilon_3 - 1/2} (-1+\underset{\sim}{x}_{i_1}^{\varepsilon_1})^{(r_1+1)} (-1+\underset{\sim}{x}_{i_2}^{\varepsilon_2})^{r_2} .$$

$$.(-1+\underset{\sim}{x}_{i_3}^{\varepsilon_3})^{r_3} (-1+\underset{\sim}{x}_{i_4}^{\varepsilon_4})^{(r_4+1)} \ldots\ldots (-1+\underset{\sim}{x}_{i_{t_2}}^{\varepsilon_{t_2}})^{(r_{t_2}+1)} .$$

A term of $x_{i_3 i_2}(C_1^{\delta_1})$ is

$$\delta_1\varepsilon_2\varepsilon_3 \underset{\sim}{x}_{i_2}^{\varepsilon_2 - 1/2} \underset{\sim}{x}_{i_3}^{\varepsilon_3 - 1/2} \underset{\sim}{x}_{i_1}^{\varepsilon_1(r_1+1)} \underset{\sim}{x}_{i_2}^{\varepsilon_2 r_2} \underset{\sim}{x}_{i_3}^{\varepsilon_3 r_3} \underset{\sim}{x}_{i_4}^{\varepsilon_4(r_4+1)} \dots \underset{\sim}{x}_{i_{t_2}}^{\varepsilon_{t_2}(r_{t_2}+1)}$$

which is the only term of length precisely $r^* - 2 + \tau$, where

$$\tau = \begin{cases} 0 & \text{if } \varepsilon_2 = \varepsilon_3 = 1 \\ 1 & \text{if } \varepsilon_2 + \varepsilon_3 = 0 \\ 2 & \text{if } \varepsilon_2 = \varepsilon_3 = -1 . \end{cases}$$

Since (as in Step I)

$$0 = x_{i_3 i_2}(w) = x_{i_3 i_2}(C_1^{\delta_1}) + \dots + x_{i_3 i_2}(C_m^{\delta_m}),$$

there is a factor (say) C^{δ} in w which is different from $C_1^{\delta_1}$ such

that $x_{i_3 i_2}(C^{\delta})$ has a term

$$-\delta_1\varepsilon_2\varepsilon_3 \underset{\sim}{x}_{i_2}^{\varepsilon_2 - 1/2} \underset{\sim}{x}_{i_3}^{\varepsilon_3 - 1/2} \underset{\sim}{x}_{i_1}^{\varepsilon_1(r_1+1)} \underset{\sim}{x}_{i_2}^{\varepsilon_2 r_2} \underset{\sim}{x}_{i_3}^{\varepsilon_3 r_3} \underset{\sim}{x}_{i_4}^{\varepsilon_4(r_4+1)} \dots \underset{\sim}{x}_{i_{t_2}}^{\varepsilon_{t_2}(r_{t_2}+1)} .$$

Thus, using 2.4.9(ii), 1.2.1, 1.4.6 and 1.4.3 I can take

$$C^{\delta} = [u_1^{-1}, x_{i_2}^{-\varepsilon_2'}; u_2, x_{i_3}^{\varepsilon_3'}, u_3, \dots, u_r]^{\delta} \quad (r \geq 2) \quad ,$$

where $\varepsilon_2', \varepsilon_3', \delta \in \{1, -1\}$ and $u_i \in XUX^{-1}$.

If $\{u_1, u_2\} = \{x_{i_2}^{\varepsilon_2''}, x_{i_3}^{\varepsilon_3''}\}$

for some $\varepsilon_2'', \varepsilon_3'' \in \{1,-1\}$, then by the assertion 2.4.14, $u_i = u_j = x_{i_1}^{\varepsilon_1}$ does not hold for any pair $u_i, u_j (i \neq j)$ and so no term of $x_{i_3 i_2}(C^\delta)$ can have $\underset{\sim}{x}_{i_1}^{\varepsilon_i(r_1+1)}$ as a factor which contradicts the choice of C^δ. Thus $\{u_1, u_2\} \neq \{x_{i_2}^{\varepsilon_2''}, x_{i_3}^{\varepsilon_3''}\}$ for any $\varepsilon_2'', \varepsilon_3'' \in \{1,-1\}$, so that by 2.4.9(ii) (or 2.4.10(i))

$$x_{i_3 i_2}(C^\delta) = \delta\varepsilon_2'\varepsilon_3' \underset{\sim}{x}_{i_2}^{\varepsilon_2' - 1/2} \underset{\sim}{x}_{i_3}^{\varepsilon_3' - 1/2} (-1+\underset{\sim}{u}_1) \ldots. (-1+\underset{\sim}{u}_r).$$

But, by hypothesis, a term of $x_{i_3 i_2}(C^\delta)$ contains $\underset{\sim}{x}_{i_1}^{\varepsilon_1(r_1+1)}$ $(r_1 \geq 1)$ as a factor so that in $\{u_1, \ldots, u_r\}$ there are $(r_1+1) x_{i_1}^{\varepsilon_1}$'s and in particular,

$$u_i = u_j = x_{i_1}^{\varepsilon_1} \text{ for some } i \neq j.$$

Thus, by 2.4.14

$$u_1 = x_{i_1}^{\varepsilon_1} \text{ and } u_2 = x_{i_1}^{\varepsilon_1}.$$

Further, if $\varepsilon_2' = -\varepsilon_2$ then $x_{i_2}^{-\varepsilon_2} \notin \{u_3, \ldots, u_r\}$ (since C is a special commutator in w) and so $\underset{\sim}{x}_{i_2}^{-\varepsilon_2 - 1/2}$ is a factor in each term of $x_{i_3 i_2}(C^\delta)$ which contradicts the choice of C^δ that a term of $x_{i_3 i_2}(C^\delta)$ contains a factor $\underset{\sim}{x}_{i_2}^{(\varepsilon_2 - 1/2) + \varepsilon_2 r_2}$ $(r_2 \geq 0)$. Thus $\varepsilon_2' = \varepsilon_2$ and similarly $\varepsilon_3' = \varepsilon_3$. This now implies that $\{u_3, \ldots, u_r\}$ contains

$(r_1-1)x_{i_1}^{\varepsilon_1}$'s, $r_2x_{i_2}^{\varepsilon_2}$'s, $r_3x_{i_3}^{\varepsilon_3}$'s, $(r_4+1)x_{i_4}^{\varepsilon_4}$'s,....,$(r_{t_2}+1)x_{i_{t_2}}^{\varepsilon_{t_2}}$'s which

gives in turn

$$r-2 \geq r^*-4; wtC = r+2 \geq r^* = wtC_1;$$

$$wtC_1 = wtC \quad (\text{since } r^* \geq r+2); \quad \text{and}$$

hence also $\delta = -\delta_1$.

Thus (using 1.4.4)

$$C^\delta = [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{\varepsilon_1}, x_{i_3}^{\varepsilon_3}, (r_1-1)x_{i_1}^{\varepsilon_1}, r_2x_{i_2}^{\varepsilon_2}, r_3x_{i_3}^{\varepsilon_3}, (r_4+1)x_{i_4}^{\varepsilon_4}, \ldots, (r_{t_2}+1)x_{i_{t_2}}^{\varepsilon_{t_2}}]^{-\delta_1}.$$

But $C_1^{\delta_1}C^\delta = e$ modulo $[F'',F]$, which contradicts the minimality of the representation of w_{12}.

Next I consider the case when

$$C_1^{\delta_1} = [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{\varepsilon_1}, x_{i_2}^{\varepsilon_2}, (r_1-1)x_{i_1}^{\varepsilon_1}, (r_2-1)x_{i_2}^{\varepsilon_2}, (r_3+1)x_{i_3}^{\varepsilon_3}, \ldots, (r_{t_2}+1)x_{i_{t_2}}^{\varepsilon_{t_2}}]^{\delta_1}.$$

By 2.4.10(ii)

$$x_{i_2 i_2}(C_1^{\delta_1}) = \delta_1 \underset{\sim}{x}_{i_2}^{\varepsilon_2-1}(-1+\underset{\sim}{x}_{i_1}^{\varepsilon_1})^{(r_1+1)}(-1+\underset{\sim}{x}_{i_2}^{\varepsilon_2})^{(r_2-1)}.$$

$$.(-1+\underset{\sim}{x}_{i_3}^{\varepsilon_3})^{(r_3+1)} \ldots . (-1+\underset{\sim}{x}_{i_{t_2}}^{\varepsilon_{t_2}})^{(r_{t_2}+1)}$$

$$-\delta_1 \underset{\sim}{x}_{i_2}^{-\varepsilon_2 - 1} (-1+\underset{\sim}{x}_{i_1}^{-\varepsilon_1})^{(r_1+1)} (-1+\underset{\sim}{x}_{i_2}^{-\varepsilon_2})^{(r_2-1)} .$$

$$.(-1+\underset{\sim}{x}_{i_3}^{-\varepsilon_3})^{(r_3+1)} \;\ldots.\; (-1+\underset{\sim}{x}_{i_{t_2}}^{-\varepsilon_{t_2}})^{(r_{t_2}+1)} \;;$$

and a term of $x_{i_2 i_2}(C_1^{\delta_1})$ is

$$\delta_1 \underset{\sim}{x}_{i_2}^{\varepsilon_2 - 1} \underset{\sim}{x}_{i_1}^{\varepsilon_1(r_1+1)} \underset{\sim}{x}_{i_2}^{\varepsilon_2(r_2-1)} \underset{\sim}{x}_{i_3}^{\varepsilon_3(r_3+1)} \;\ldots.\; \underset{\sim}{x}_{i_{t_2}}^{\varepsilon_{t_2}(r_{t_2}+1)}$$

which is of length precisely $r^* - 2 + \tau$, where

$$\tau = \begin{cases} 0 & \text{if } \varepsilon_2 = 1 \\ 2 & \text{if } \varepsilon_2 = -1 \;; \end{cases}$$

and this does not cancel off with any other term of $x_{i_2 i_2}(C_1^{\delta_1})$.

Since

$$0 = x_{i_2 i_2}(w) = x_{i_2 i_2}(C_1^{\delta_1}) + \ldots + x_{i_2 i_2}(C_m^{\delta_m}),$$

it follows that for some factor C^{δ} different from $C_1^{\delta_1}$, a term of $x_{i_2 i_2}(C^{\delta})$ is

$$-\delta_1 \underset{\sim}{x}_{i_2}^{\varepsilon_2 - 1} \underset{\sim}{x}_{i_1}^{\varepsilon_1(r_1+1)} \underset{\sim}{x}_{i_2}^{\varepsilon_2(r_2-1)} \underset{\sim}{x}_{i_3}^{\varepsilon_3(r_3+1)} \;\ldots.\; \underset{\sim}{x}_{i_{t_2}}^{\varepsilon_{t_2}(r_{t_2}+1)} \qquad (r_1, r_2 \geq 1).$$

By using 2.4.9(i), 1.2.1, 1.4.6 and 1.4.3, I can take

$$C^{\delta} = [u_1^{-1}, x_{i_2}^{-\varepsilon_2'}; u_2, x_{i_2}^{\varepsilon_2''}, u_3, \ldots, u_r]^{\delta} .$$

By 2.4.10(i) or (ii)

$$x_{i_2i_2}(C^\delta) = \delta\varepsilon_2'\varepsilon_2''\underset{\sim}{x}_{i_2}^{\varepsilon_2'-1/2}\underset{\sim}{x}_{i_2}^{\varepsilon_2''-1/2}(-1+\underset{\sim}{u}_1) \ldots\ldots (-1+\underset{\sim}{u}_r)$$

$$-\delta\varepsilon_2'\varepsilon_2''\underset{\sim}{x}_{i_2}^{-\varepsilon_2'-1/2}\underset{\sim}{x}_{i_2}^{-\varepsilon_2''-1/2}(-1+\underset{\sim}{u}_1^{-1}) \ldots\ldots (-1+\underset{\sim}{u}_r^{-1}).$$

By hypothesis, a term of $x_{i_2i_2}(C^\delta)$ contains $\underset{\sim}{x}_{i_1}^{\varepsilon_1(r_1+1)}$ $(r_1 \geq 1)$ as a factor, which implies that either in $\{u_1,\ldots,u_r\}$ or in $\{u_1^{-1},\ldots,u_r^{-1}\}$ there are (r_1+1) $x_{i_1}^{\varepsilon_1}$'s. Thus by 2.4.14, for some $\varepsilon_1' \in \{1,-1\}$

$$u_1 = x_{i_1}^{\varepsilon_1'} \text{ and } u_2 = x_{i_1}^{\varepsilon_1'}.$$

If $\varepsilon_1' = \varepsilon_1$, then I need consider only the terms of the first half of $x_{i_2i_2}(C^\delta)$, because, no term of the second half of $x_{i_2i_2}(C^\delta)$ can contain $\underset{\sim}{x}_{i_1}^{\varepsilon_1}$ as a factor (since C is special).

But then $\varepsilon_2' = \varepsilon_2''$, for otherwise, each term of the first half of $x_{i_2i_2}(C^\delta)$ contains $\underset{\sim}{x}_{i_2}^{-1}$ as a factor (since C is special) which contradicts the assumption that a term of the first half of $x_{i_2i_2}(C^\delta)$ contains $\underset{\sim}{x}_{i_2}^{(\varepsilon_2-1)+(r_2-1)\varepsilon_2}$ $(r_2 \geq 1)$.

Further $\varepsilon_2' = \varepsilon_2$, for otherwise each term of the first half of $x_{i_2 i_2}(C^\delta)$ contains $\underset{\sim}{x}_{i_2}^{-\varepsilon_2-1}$ as a factor, but by hypothesis, a term of the first half of $x_{i_2 i_2}(C^\delta)$ contains $\underset{\sim}{x}_{i_2}^{\varepsilon_2-1+(r_2-1)\varepsilon_2}$ $(r_2 \geq 1)$, which is a contradiction.

Thus, it follows that $\{u_3,\dots,u_r\}$ contains $(r_1-1)x_{i_1}^{\varepsilon_1}$'s, $(r_2-1)x_{i_2}^{\varepsilon_2}$'s, $(r_3+1)x_{i_3}^{\varepsilon_3}$'s, ..., $(r_{t_2}+1)x_{i_{t_2}}^{\varepsilon_{t_2}}$'s which gives that

$$r-2 \geq r^*-4; \quad wtC = r+2 \geq r^* = wtC_1;$$

$$wtC = wtC_1 \quad (\text{since } r^* \geq r+2);$$

and therefore $\delta = -\delta_1$. Hence (by using 1.4.4)

$$C^\delta = [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{\varepsilon_1}, x_{i_2}^{\varepsilon_2}, (r_1-1)x_{i_1}^{\varepsilon_1}, (r_2-1)x_{i_2}^{\varepsilon_2}, (r_3+1)x_{i_3}^{\varepsilon_3}, \dots, (r_{t_2}+1)x_{i_{t_2}}^{\varepsilon_{t_2}}]^{-\delta_1}.$$

Thus $C_1^{\delta_1}C^\delta = e$ modulo $[F'',F]$, which gives contradiction to the minimal representation of w_{12}.

If $\varepsilon_1' = -\varepsilon_1$, then no term of the first half of $x_{i_2 i_2}(C^\delta)$ can contain $\underset{\sim}{x}_{i_1}^{\varepsilon_1(r_1+1)}$ $(r_1 \geq 1)$ (since C is special), thus I need only consider terms of the second half of $x_{i_2 i_2}(C^\delta)$. But then it implies that $\varepsilon_2' = \varepsilon_2'' = -\varepsilon_2$, for otherwise each term of the second half of

$x_{i_2 i_2}(C^{\delta})$ contains $\underset{\sim}{x}_{i_2}^{-\varepsilon_2 - 1}$ as a factor (since C is special), which is a contradiction to the fact that a term of $x_{i_2 i_2}(C^{\delta})$ contains $\underset{\sim}{x}_{i_2}^{(\varepsilon_2 - 1)+(r_2 - 1)\varepsilon_2}$ $(r_2 \geq 1)$ as a factor. Thus it follows that $\{u_3^{-1}, \ldots, u_r^{-1}\}$ contains $(r_1 - 1)x_{i_1}^{\varepsilon_1}$'s, $(r_2 - 1)x_{i_2}^{\varepsilon_2}$'s, $(r_3 + 1)x_{i_3}^{\varepsilon_3}$'s, ..., $(r_{t_2} + 1)x_{i_{t_2}}^{\varepsilon_{t_2}}$'s which gives that $wtC = wtC_1$ and therefore, it follows that $\delta = \delta_1$. Hence

$$C^{\delta} = [x_{i_1}^{\varepsilon_1}, x_{i_2}^{\varepsilon_2}; x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}, u_3^{-1}, \ldots, u_r^{-1}]^{\delta_1}$$

$$= [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{\varepsilon_1}, x_{i_2}^{\varepsilon_2}, u_3, \ldots, u_r]^{-\delta_1} \quad \text{by } 1.2.1, 1.4.6, 1.4.4.$$

But $C_1^{\delta_1} C^{\delta} = e$ modulo $[F'', F]$, which again contradicts the choice of w_{12}. This completes the proof that w_{12} is empty.

DETAILS OF STEP III

By previous steps, w_{11} and w_{12} are both empty. Here I assume that w_{13} is non-empty, so that $C_1^{\delta_1}$ is a factor of w_{13}.

Since $C_1 \in$ Cat. III, I write

$$C_1^{\delta_1} = [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{-\varepsilon_1}, x_{i_3}^{\varepsilon_3}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{\delta_1} \quad (t_3 \geq 3,\ \delta_1 \in \{1,-1\})$$

where $x_{i_j} \neq x_{i_k}$ for $j \neq k$ and $t_3 + 1 = r^*$.

About the remaining factors of w_{13}, by 2.4.7, I can make the following assertion:

2.4.15 If $[v_1^{-1}, v_2^{-1}; v_3, v_4, \ldots, v_{s_3}]^{\delta}$ $(s_3 \geq 4,\ \delta \in \{1,-1\})$ is a factor of w_{13} such that

$$x_{i_2}^{\varepsilon_2'}, x_{i_3}^{\varepsilon_3'} \in \{v_1, v_2, \ldots, v_{s_3}\} \quad \text{for some}$$

$\varepsilon_2', \varepsilon_3' \in \{1,-1\}$ then

either $x_{i_2}^{\varepsilon_2'} \in \{v_1, v_2\}$ and $x_{i_3}^{\varepsilon_3'} \in \{v_3, v_4\}$

or $x_{i_3}^{\varepsilon_3'} \in \{v_1, v_2\}$ and $x_{i_2}^{\varepsilon_2'} \in \{v_3, v_4\}$ (for $s_3 = 4$, the assertion 2.4.15 is obvious).

There is a representation of w as a power product of special commutators in which w_{11}, w_{12} are empty and the factors of w_{13} satisfy 2.4.15. Among all such representations of w I choose one in which w_{13} consists of least number of factors and I write

$$w_{13} = C_{13}^{\delta_{13}} C_{23}^{\delta_{23}} \ldots . C_{m_3 3}^{\delta_{m_3 3}} \quad (m_3 \geq 1,\ \delta_{13}, \ldots, \delta_{m_3 3} \in \{1,-1\}),$$

where $C_{13}^{\delta_{13}} = C_1^{\delta_1}$.

Now, I am in a position to give details of Step III.

$$C_1^{\delta_1} = [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{-\varepsilon_1}, x_{i_3}^{\varepsilon_3}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{\delta_1} \quad (t_3 \geq 3).$$

By 2.4.10(i)

$$x_{i_1 i_1}(C_1^{\delta_1}) = -\delta_1 \underset{\sim}{x}_{i_1}^{\varepsilon_1 - 1/2} \underset{\sim}{x}_{i_1}^{-\varepsilon_1 - 1/2}(-1+\underset{\sim}{x}_{i_2}^{\varepsilon_2}) \ldots (-1+\underset{\sim}{x}_{i_{t_3}}^{\varepsilon_{t_3}})$$

$$+ \delta_1 \underset{\sim}{x}_{i_1}^{-\varepsilon_1 - 1/2} \underset{\sim}{x}_{i_1}^{\varepsilon_1 - 1/2}(-1+\underset{\sim}{x}_{i_2}^{-\varepsilon_2}) \ldots (-1+\underset{\sim}{x}_{i_{t_3}}^{-\varepsilon_{t_3}}).$$

A term of $x_{i_1 i_1}(C_1^{\delta_1})$ is

$$-\delta_1 \underset{\sim}{x}_{i_1}^{-1} \underset{\sim}{x}_{i_2}^{\varepsilon_2} \ldots \underset{\sim}{x}_{i_{t_3}}^{\varepsilon_{t_3}}$$

which is of length precisely t_3 (because C_1 is special).

Since (as before)

$$0 = x_{i_1 i_1}(w) = x_{i_1 i_1}(C_1^{\delta_1}) + \ldots + x_{i_1 i_1}(C_m^{\delta_m}),$$

it follows that for some factor (say) C^{δ} in w different from $C_1^{\delta_1}$, a term of $x_{i_1 i_1}(C^{\delta})$ is

$$+\delta_1 \underset{\sim}{x}_{i_1}^{-1} \underset{\sim}{x}_{i_2}^{\varepsilon_2} \ldots \underset{\sim}{x}_{i_{t_3}}^{\varepsilon_{t_3}}.$$

By 2.4.9(i), 1.2.1, 1.4.6 and 1.4.3 I can write

$$C^{\delta} = [x_{i_1}^{-\varepsilon_1'}, u_1^{-1}; x_{i_1}^{\varepsilon_1''}, u_2, \ldots, u_r]^{\delta} \quad (r \geq 2,\ u_i \in XUX^{-1}) ,$$

where $r + 2 \leq r^*$ by 2.4.1 and $\varepsilon_1', \varepsilon_1'' \in \{1,-1\}$. By 2.4.10(i) or (ii)

$$x_{i_1 i_1}(C^\delta) = \delta\varepsilon_1'\varepsilon_1'' x_{i_1}^{\varepsilon_1' - 1/2} x_{i_1}^{\varepsilon_1'' - 1/2}(-1+\underset{\sim}{u}_1) \ldots . (-1+\underset{\sim}{u}_t)$$

$$-\delta\varepsilon_1'\varepsilon_1'' \underset{\sim}{x}_{i_1}^{-\varepsilon_1' - 1/2} \underset{\sim}{x}_{i_1}^{-\varepsilon_1'' - 1/2}(-1+\underset{\sim}{u}_1^{-1}) \ldots . (-1+\underset{\sim}{u}_r^{-1}).$$

It follows that

either $\left\{x_{i_2}^{\varepsilon_2}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}\right\} \subseteq \{u_1, \ldots, u_r\}$

or $\left\{x_{i_2}^{\varepsilon_2}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}\right\} \subseteq \{u_1^{-1}, \ldots, u_r^{-1}\}$

(because $\{x_{i_1}, x_{i_1}^{-1}\} \not\subseteq \left\{x_{i_2}^{\varepsilon_2}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}\right\}$).

In each case, I have in turn

$$r \geq t_3 - 1;\ r + 2 \geq t_3 + 1;\ r + 2 = t_3 + 1$$

(since $t_3 + 1 \geq r + 2$); and it follows that

either $\left\{x_{i_2}^{\varepsilon_2}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}\right\} = \{u_1, \ldots, u_r\}$

or $\left\{x_{i_2}^{\varepsilon_2}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}\right\} = \{u_1^{-1}, \ldots, u_r^{-1}\}$.

This now gives that $\varepsilon_1'' = -\varepsilon_1'$; and either $\delta = -\delta_1$, so that

$$C^{\delta} = [x_{i_1}^{-\varepsilon_1'}, x_{i_{2\sigma}}^{-\varepsilon_{2\sigma}}; x_{i_1}^{-\varepsilon_1'}, x_{i_{3\sigma}}^{\varepsilon_{3\sigma}}, \ldots, x_{i_{t_{3\sigma}}}^{\varepsilon_{t_{3\sigma}}}]^{-\delta_1}$$

$$= [x_{i_1}^{-\varepsilon_1}, x_{i_{2\sigma}}^{\varepsilon_{2\sigma}}; x_{i_1}^{-\varepsilon_1}, x_{i_{3\sigma}}^{\varepsilon_{3\sigma}}, \ldots, x_{i_{t_{3\sigma}}}^{\varepsilon_{t_{3\sigma}}}]^{-\delta_1} \qquad \text{by } 1.4.8,$$

or $\delta = \delta_1$, so that

$$C^{\delta} = [x_{i_1}^{-\varepsilon_1'}, x_{i_{2\sigma}}^{\varepsilon_{2\sigma}}; x_{i_1}^{-\varepsilon_1'}, x_{i_{3\sigma}}^{-\varepsilon_{3\sigma}}, x_{i_{4\sigma}}^{-\varepsilon_{4\sigma}}, \ldots, x_{i_{t_{3\sigma}}}^{-\varepsilon_{t_{3\sigma}}}]^{\delta_1}$$

$$= [x_{i_1}^{-\varepsilon_1}, x_{i_{3\sigma}}^{-\varepsilon_{3\sigma}}; x_{i_1}^{-\varepsilon_1}, x_{i_{2\sigma}}^{\varepsilon_{2\sigma}}, x_{i_4}^{\varepsilon_{4\sigma}}, \ldots, x_{i_{t_{3\sigma}}}^{\varepsilon_{t_{3\sigma}}}]^{-\delta_1}$$

by 1.4.8, 1.2.1, 1.4.6 and 1.4.4,

where σ is a permutation of $\{2,3,\ldots,t_3\}$. Now, by the assertion 2.4.15,

either $$C^{\delta} = [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{-\varepsilon_1}, x_{i_3}^{\varepsilon_3}, x_{i_{4\sigma}}^{\varepsilon_{4\sigma}}, \ldots, x_{i_{t_{3\sigma}}}^{\varepsilon_{t_{3\sigma}}}]^{-\delta_1}$$

$$= [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{-\varepsilon_1}, x_{i_3}^{\varepsilon_3}, x_{i_4}^{\varepsilon_4}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{-\delta_1} \qquad \text{by } 1.4.4,$$

in which case $C_1^{\delta_1} C^{\delta} = e$ modulo $[F'', F]$, which is contrary to the choice of w_{13},

or $$C^{\delta} = [x_{i_1}^{-\varepsilon_1}, x_{i_3}^{-\varepsilon_3}; x_{i_1}^{-\varepsilon_1}, x_{i_2}^{\varepsilon_2}, x_{i_4}^{\varepsilon_4}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{-\delta_1}$$

in which case (when $t_3 \geq 4$)

$$c_1^{\delta_1}c^{\delta} = [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{-\varepsilon_1}, x_{i_3}^{\varepsilon_3}, x_{i_4}^{\varepsilon_4}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{\delta_1}.$$

$$. [x_{i_1}^{-\varepsilon_1}, x_{i_3}^{-\varepsilon_3}; x_{i_1}^{-\varepsilon_1}, x_{i_2}^{\varepsilon_2}, x_{i_4}^{\varepsilon_4}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{-\delta_1}$$

$$= [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}, x_{i_4}^{-\varepsilon_4}; x_{i_1}^{-\varepsilon_1}, x_{i_3}^{\varepsilon_3}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{\delta_1}.$$

$$. [x_{i_1}^{-\varepsilon_1}, x_{i_3}^{-\varepsilon_3}, x_{i_4}^{-\varepsilon_4}; x_{i_1}^{-\varepsilon_1}, x_{i_2}^{\varepsilon_2}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{-\delta_1} \quad \text{by } 1.4.6$$

$$= [x_{i_1}^{-\varepsilon_1}, x_{i_4}^{-\varepsilon_4}; x_{i_1}^{-\varepsilon_1}, x_{i_3}^{\varepsilon_3}, x_{i_2}^{\varepsilon_2}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{\delta_1}.$$

$$.[x_{i_4}^{-\varepsilon_4}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{-\varepsilon_1}, x_{i_3}^{\varepsilon_3}, x_{i_1}^{\varepsilon_1}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{\delta_1}$$

$$.[x_{i_1}^{-\varepsilon_1}, x_{i_4}^{-\varepsilon_4}; x_{i_1}^{-\varepsilon_1}, x_{i_2}^{\varepsilon_2}, x_{i_3}^{\varepsilon_3}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{-\delta_1}.$$

$$.[x_{i_4}^{-\varepsilon_4}, x_{i_3}^{-\varepsilon_3}; x_{i_1}^{-\varepsilon_1}, x_{i_2}^{\varepsilon_2}, x_{i_1}^{\varepsilon_1}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{-\delta_1}$$

by 1.4.5, 1.4.6, 1.4.4

$$= [x_{i_1}^{-\varepsilon_1}, x_{i_4}^{-\varepsilon_4}; x_{i_2}^{\varepsilon_2}, x_{i_3}^{\varepsilon_3}, x_{i_1}^{-\varepsilon_1}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{\delta_1}.$$

$$.[x_{i_4}^{-\varepsilon_4}, x_{i_2}^{-\varepsilon_2}; x_{i_3}^{\varepsilon_3}, x_{i_1}^{\varepsilon_1}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{\delta_1}.$$

$$.[x_{i_4}^{-\varepsilon_4}, x_{i_2}^{-\varepsilon_2}; x_{i_3}^{\varepsilon_3}, x_{i_1}^{-\varepsilon_1}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{\delta_1}.$$

$$.[x_{i_4}^{-\varepsilon_4}, x_{i_3}^{-\varepsilon_3}; x_{i_2}^{\varepsilon_2}, x_{i_1}^{\varepsilon_1}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{-\delta_1}.$$

$$.[x_{i_4}^{-\varepsilon_4}, x_{i_3}^{-\varepsilon_3}; x_{i_2}^{\varepsilon_2}, x_{i_1}^{-\varepsilon_1}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{-\delta_1} \quad \text{by } 1.4.5, 1.4.3, 1.4.7$$

$$= [x_{i_4}^{-\varepsilon_4}, x_{i_1}^{\varepsilon_1}; x_{i_2}^{\varepsilon_2}, x_{i_3}^{\varepsilon_3}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{\delta_1}.$$

$$.[x_{i_4}^{-\varepsilon_4}, x_{i_1}^{-\varepsilon_1}; x_{i_2}^{\varepsilon_2}, x_{i_3}^{\varepsilon_3}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{\delta_1}.$$

$$.[x_{i_4}^{-\varepsilon_4}, x_{i_2}^{-\varepsilon_2}; x_{i_3}^{\varepsilon_3}, x_{i_1}^{\varepsilon_1}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{\delta_1}.$$

$$.[x_{i_4}^{-\varepsilon_4}, x_{i_2}^{-\varepsilon_2}; x_{i_3}^{\varepsilon_3}, x_{i_1}^{-\varepsilon_1}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{\delta_1}.$$

$$.[x_{i_4}^{-\varepsilon_4}, x_{i_3}^{-\varepsilon_3}; x_{i_2}^{\varepsilon_2}, x_{i_1}^{\varepsilon_1}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{-\delta_1}.$$

$$.[x_{i_4}^{-\varepsilon_4}, x_{i_3}^{-\varepsilon_3}; x_{i_2}^{\varepsilon_2}, x_{i_1}^{-\varepsilon_1}, \ldots, x_{i_{t_3}}^{\varepsilon_{t_3}}]^{-\delta_1} \quad \text{by } 1.4.4, 1.4.6, 1.4.7$$

which is a power product of special commutators of weight strictly less than r^* and hence gives a representation of w_{13} with fewer factors — contrary to assumption. Thus, it remains to consider the case $t_3 = 3$.

The case $t_3 = 3$ is different and requires special argument.

Let

$$C_1^{\delta_1} = [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{-\varepsilon_1}, x_{i_3}^{\varepsilon_3}]^{\delta_1}.$$

Then

$$x_{i_1 i_2}(C_1^{\delta_1}) = \delta_1 \varepsilon_1 \varepsilon_2 \underset{\sim}{x}_{i_2}^{\varepsilon_2 - 1/2} \underset{\sim}{x}_{i_1}^{-\varepsilon_1 - 1/2} (-1 + \underset{\sim}{x}_{i_1}^{\varepsilon_1})(-1 + \underset{\sim}{x}_{i_3}^{\varepsilon_3}).$$

A term of $x_{i_1 i_2}(C_1^{\delta_1})$ is

$$\delta_1 \varepsilon_1 \varepsilon_2 \underset{\sim}{x}_{i_1}^{\varepsilon_1 - 1/2} \underset{\sim}{x}_{i_2}^{\varepsilon_2 - 1/2} \underset{\sim}{x}_{i_3}^{\varepsilon_3};$$

which does not cancel off with any other term of $x_{i_1 i_2}(C_1^{\delta_1})$. Thus (as before) there is a factor (say) C^{δ} in w such that a term of $x_{i_1 i_2}(C^{\delta})$ is

$$-\delta_1 \varepsilon_1 \varepsilon_2 \underset{\sim}{x}_{i_1}^{\varepsilon_1 - 1/2} \underset{\sim}{x}_{i_2}^{\varepsilon_2 - 1/2} \underset{\sim}{x}_{i_3}^{\varepsilon_3}.$$

Since $t_3 + 1 = r^* = 4$, it follows that weight of C^{δ} is 4 and that $C^{\delta} \in$ Cat.III. I write

$$C^{\delta} = [u_1^{-1}, x_{i_2}^{-\varepsilon_2'}; x_{i_1}^{-\varepsilon_1'}, u_2]^{\delta} \quad (\varepsilon_1', \varepsilon_2' \in \{1, -1\}).$$

By 2.4.10(i)

$$x_{i_1 i_2}(C^{\delta}) = \delta \varepsilon_1' \varepsilon_2' \underset{\sim}{x}_{i_1}^{-\varepsilon_1' - 1/2} \underset{\sim}{x}_{i_2}^{\varepsilon_2' - 1/2} (-1 + \underset{\sim}{u}_1)(-1 + \underset{\sim}{u}_2).$$

Since by hypothesis, a term of $x_{i_1 i_2}(C^{\delta})$ is

$$-\delta_1 \varepsilon_1 \varepsilon_2 \underset{\sim}{x}_{i_1}^{\varepsilon_1 - 1/2} \underset{\sim}{x}_{i_2}^{\varepsilon_2 - 1/2} \underset{\sim}{x}_{i_3}^{\varepsilon_3},$$

it follows that

either

$$C^{\delta} = [x_{i_3}^{-\varepsilon_3}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{\varepsilon_1}, x_{i_3}^{-\varepsilon_3}]^{-\delta_1},$$

or

$$C^{\delta} = [x_{i_3}^{-\varepsilon_3}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{\varepsilon_1}, x_{i_2}^{-\varepsilon_2}]^{-\delta_1} .$$

Thus, considering $x_{i_1 i_2}(C_1^{\delta_1})$ shows that w_{13} also contains at least one of the two factors

$$[x_{i_3}^{-\varepsilon_3}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{\varepsilon_1}, x_{i_3}^{-\varepsilon_3}]^{-\delta_1} ,$$

$$[x_{i_3}^{-\varepsilon_3}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{\varepsilon_1}, x_{i_2}^{-\varepsilon_2}]^{-\delta_1} .$$

Applying similar argument to either of the above factors implies the existence of at least one more factor in w_{13}. At most nine further applications of the similar argument ensure that w_{13} contains a power product (say)

$$C_1^{\delta_1} C_2^{\delta_2} C_3^{\delta_3}$$

or

$$C_1^{\delta_1} C_2^{\delta_2} C_4^{\delta_4}$$

where

$$C_2^{\delta_2} = C_{23}^{\delta_{23}} = [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{\varepsilon_2}; x_{i_1}^{-\varepsilon_1}, x_{i_3}^{\varepsilon_3}]^{\delta_1}$$

$$C_3^{\delta_3} = C_{33}^{\delta_{33}} = [x_{i_3}^{-\varepsilon_3'}, x_{i_2}^{-\varepsilon_2}; x_{i_2}^{-\varepsilon_2}, x_{i_1}^{\varepsilon_1'}]^{\delta_1}$$

$$C_4^{\delta_4} = C_{43}^{\delta_{43}} = [x_{i_1}^{-\varepsilon_1''}, x_{i_3}^{-\varepsilon_3}; x_{i_3}^{-\varepsilon_3}, x_{i_2}^{\varepsilon_2''}]^{\delta_1}$$

for some $\varepsilon_1', \varepsilon_2'', \varepsilon_1'', \varepsilon_3' \in \{1,-1\}$

Now considering $x_{i_1 i_1}(C_1^{\delta_1})$, $x_{i_1 i_1}(C_2^{\delta_2})$, $x_{i_2 i_2}(C_3^{\delta_3})$ and $x_{i_3 i_3}(C_4^{\delta_4})$ implies that w_{13} contains a power product (say)

$$C_1^{\delta_1} C_2^{\delta_2} C_3^{\delta_3} C_5^{\delta_5} C_6^{\delta_6} C_7^{\delta_7}$$

or

$$C_1^{\delta_1} C_2^{\delta_2} C_4^{\delta_4} C_5^{\delta_5} C_6^{\delta_6} C_8^{\delta_8},$$

where

$$C_5^{\delta_5} = [x_{i_1}^{-\varepsilon_1}, x_{i_3}^{-\varepsilon_3}; x_{i_1}^{-\varepsilon_1}, x_{i_2}^{\varepsilon_2}]^{-\delta_1}$$

$$C_6^{\delta_6} = [x_{i_1}^{-\varepsilon_1}, x_{i_3}^{-\varepsilon_3}; x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}]^{-\delta_1}$$

$$C_7^{\delta_7} = [x_{i_1}^{-\varepsilon_1'}, x_{i_2}^{-\varepsilon_2}; x_{i_2}^{-\varepsilon_2}, x_{i_3}^{\varepsilon_3'}]^{-\delta_1} \quad \text{and}$$

$$C_8^{\delta_8} = [x_{i_2}^{-\varepsilon_2''}, x_{i_3}^{-\varepsilon_3}; x_{i_3}^{-\varepsilon_3}, x_{i_1}^{\varepsilon_1''}]^{-\delta_1}$$ (see pages 69-71).

By 1.4.11

$$C_1^{\delta_1} C_2^{\delta_2} C_3^{\delta_3} C_5^{\delta_5} C_6^{\delta_6} C_7^{\delta_7} = [x_{i_3}^{-\varepsilon_3'}, x_{i_2}^{-\varepsilon_2}; x_{i_2}^{-\varepsilon_2}, x_{i_1}^{-\varepsilon_1'}]^{-\delta_1}.$$

$$.[x_{i_3}^{\varepsilon_3'}, x_{i_2}^{-\varepsilon_2}; x_{i_2}^{-\varepsilon_2}, x_{i_1}^{\varepsilon_1'}]^{-\delta_1};$$

and also

$$c_1^{\delta_1} c_2^{\delta_2} c_4^{\delta_4} c_5^{\delta_5} c_6^{\delta_6} c_8^{\delta_8} = [x_{i_1}^{-\varepsilon_1''}, x_{i_3}^{-\varepsilon_3}; x_{i_3}^{-\varepsilon_3}, x_{i_2}^{-\varepsilon_2''}]^{-\delta_1} \, .$$

$$\cdot [x_{i_1}^{\varepsilon_1''}, x_{i_3}^{-\varepsilon_3}; x_{i_3}^{-\varepsilon_3}, x_{i_2}^{\varepsilon_2''}]^{-\delta_1} \, .$$

Thus, the power product has been replaced by a power product with fewer factors of w_{13} contrary to the choice of w_{13}. This completes the proof that w_{13} is empty.

DETAILS OF STEP IV

I have shown in the previous steps that w_{11}, w_{12} and w_{13} are empty, so that

$$w = w_{14} \, w_2 \, \hat{w} \, .$$

Now I suppose that w_{14} is non-empty and arrive at a contradiction.

First of all, I prove

2.4.16 If $C = [v_1^{-1}, v_2^{-1}; v_1^{-1}, v_3, v_4, \ldots, v_{s_4}]$ $(s_4 \geq 4,\ v_i \in XUX^{-1})$ is a special commutator such that

$$v_i = v_j = v \quad \text{for some} \quad i \neq j,$$

then C can be written as

$$C = [v_1^{-1}, v^{-1}; v_1^{-1}, v, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{s_4}]\pi,$$

where π is a power product of commutators of weight strictly less than $s_4 + 1$.

Proof. Suppose first that $v_2 = v$, Then if $v_3 = v$, there is nothing to prove. Thus $v_i = v$ for some $i > 3$, and by 1.4.4, 1.4.5

$$C = [v_1^{-1}, v^{-1}; v_1^{-1}, v, v_3, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{s_4}].$$

$$.[v_1^{-1}v; v, v_3, v_1^{-1}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{s_4}]$$

$$= [v_1^{-1}, v^{-1}; v_1^{-1}, v, v_3 \ldots, v_{i-1}, v_{i+1}, \ldots, v_{s_4}].$$

$$.[v, v_1^{-1}; v, v_3, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{s_4}].$$

$$.[v, v_1; v, v_3, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{s_4}]$$

by 1.4.4, 1.4.6, 1.4.3, 1.4.7,

where the last two commutators are of weight less than $s_4 + 1$.

If $v_3 = v$ and $v_i = v$ for some $i > 3$, the proof is similar.

Now, if $v_2 \neq v$ and $v_3 \neq v$, then by hypothesis $v_i = v_j = v$ for some i, j where $4 \leq i < j$, so that

$$C = [v_1^{-1}, v_2^{-1}; v_1^{-1}, v, v_3, v_4, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v, v_{j+1}, \ldots, v_{s_4}].$$

$$.[v_1^{-1}, v_2^{-1}; v, v_3, v_1^{-1}, v_4, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v, v_{j+1}, \ldots, v_{s_4}]$$

by 1.4.4, 1.4.5

$$= [v_1^{-1}, v_2^{-1}; v_1^{-1}, v, v_3, v_4, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v, v_{j+1}, \ldots, v_{s_4}].$$

$$.[v_2^{-1}, v_1^{-1}; v, v_3, v_4 \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v, v_{j+1}, \ldots, v_{s_4}].$$

$$.[v_2^{-1},v_1;v,v_3,v_4,\ldots,v_{i-1},v_{i+1},\ldots,v_{j-1},v,v_{j+1},\ldots,v_{s_4}]$$

by 1.4.4, 1.4.6, 1.4.3, 1.4.7,

where the last two commutators are of weight less than $s_4 + 1$; and the rest of the proof comes from the first two cases. ○

Since $C_1 \in$ Cat. IV, by definition the hypothesis of 2.4.16 is satisfied; so I can take

$$C_1^{\delta_1} = [x_{i_1}^{-\varepsilon_1},x_{i_2}^{-\varepsilon_2};x_{i_1}^{-\varepsilon_1},x_{i_2}^{\varepsilon_2},(r_2-1)x_{i_2}^{\varepsilon_2},(r_3+1)x_{i_3}^{\varepsilon_3},\ldots,(r_{t_4}+1)x_{i_{t_4}}^{\varepsilon_{t_4}}]^{\delta_1},$$

where $x_{i_j} \neq x_{i_k}$ for $j \neq k$, $r_2 \geq 1$, $r_j \geq 0$ for $j \neq 2$, $t_4 \geq 2$ and $\sum_{j=2}^{t_4} (r_j+1) + 2 = r^*$.

About the remaining factors of w, by 2.4.7, I can make the following assertion:

2.4.17 If $[v_1^{-1},v_2^{-1};v_3,v_4,\ldots,v_{s_4}]^{\delta}$ $(s_4 \geq 4)$ is a factor of w such that $\{v_1,\ldots,v_{s_4}\}$ contains $x_{i_2}^{\varepsilon_2'},x_{i_2}^{\varepsilon_2''}$ for some $\varepsilon_2',\varepsilon_2'' \in \{1,-1\}$ then

either $\quad x_{i_2}^{\varepsilon_2'} \in \{v_1,v_2\}$ and $x_{i_2}^{\varepsilon_2''} \in \{v_3,v_4\}$

or $\quad x_{i_2}^{\varepsilon_2''} \in \{v_1,v_2\}$ and $x_{i_2}^{\varepsilon_2'} \in \{v_3,v_4\}$

(the case $s_4 = 4$ does not require 2.4.7).

There is a representation of w as a power product of special commutators in which w_{11}, w_{12}, w_{13} are empty and the factors of w_{14}, w_2 satisfy 2.4.17. I take such a representation of w in which w_{14} consists of least number of factors and I write

$$w_{14} = C_{14}^{\delta_{14}} C_{24}^{\delta_{24}} \dots C_{m_4 4}^{\delta_{m_4 4}} \quad (m_4 \geq 1; \ \delta_{14},\dots,\delta_{m_4 4} \in \{1,-1\})$$

where $C_{14}^{\delta_{14}} = C_1^{\delta_1}$.

Now, I am in a position to give details of Step IV.

$$C_1^{\delta_1} = [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{-\varepsilon_1}, x_{i_2}^{\varepsilon_2}, (r_2-1)x_{i_2}^{\varepsilon_2}, (r_3+1)x_{i_3}^{\varepsilon_3}, \dots, (r_{t_4}+1)x_{i_{t_4}}^{\varepsilon_{t_4}}]^{\delta_1},$$

where $t_4 \geq 2$.

By 2.4.10(ii)

$$x_{i_1 i_1}(C_1^{\delta_1}) = -\delta_1 \underset{\sim}{x}_{i_1}^{\varepsilon_1 - 1/2} \underset{\sim}{x}_{i_1}^{-\varepsilon_1 - 1/2} (-1+\underset{\sim}{x}_{i_2}^{\varepsilon_2})^{(r_2+1)} \dots (-1+\underset{\sim}{x}_{i_{t_4}}^{\varepsilon_{t_4}})^{(r_{t_4}+1)}$$

$$+ \delta_1 \underset{\sim}{x}_{i_1}^{-\varepsilon_1 - 1/2} \underset{\sim}{x}_{i_1}^{\varepsilon_1 - 1/2} (-1+\underset{\sim}{x}_{i_2}^{-\varepsilon_2})^{(r_2+1)} \dots (-1+\underset{\sim}{x}_{i_{t_4}}^{-\varepsilon_{t_4}})^{(r_{t_4}+1)}.$$

A term of $x_{i_1 i_1}(C_1^{\delta_1})$ is

$$-\delta_1 \underset{\sim}{x}_{i_1}^{-1} \underset{\sim}{x}_{i_2}^{\varepsilon_2(r_2+1)} \dots \underset{\sim}{x}_{i_{t_4}}^{\varepsilon_{t_4}(r_{t_4}+1)};$$

and (as before) there is a factor (say) C^{δ} in w (different from $C_1^{\delta_1}$) such that a term of $x_{i_1 i_1}(C^{\delta})$ is

$$+\ \delta_1 \underset{\sim}{x}_{i_1}^{-1} \underset{\sim}{x}_{i_2}^{\varepsilon_2(r_2+1)} \ldots\ldots \underset{\sim}{x}_{i_{t_4}}^{\varepsilon_{t_4}(r_{t_4}+1)}$$

and I can write

$$C^{\delta} = [x_{i_1}^{-\varepsilon_1'}, u_1^{-1}; x_{i_1}^{\varepsilon_1''}, u_2, \ldots, u_r]^{\delta} \quad (r \geq 2,\ u_i \in XUX^{-1}),$$

where $r + 2 \leq r^*$ by 2.4.1.

By 2.4.10(ii)

$$x_{i_1 i_1}(C^{\delta}) = \delta\varepsilon_1'\varepsilon_1''\underset{\sim}{x}_{i_1}^{\varepsilon_1'-1/2}\underset{\sim}{x}_{i_1}^{\varepsilon_1''-1/2}(-1+\underset{\sim}{u}_1) \ldots\ldots (-1+\underset{\sim}{u}_r)$$

$$-\delta\varepsilon_1'\varepsilon_1''\underset{\sim}{x}_{i_1}^{-\varepsilon_1'-1/2}\underset{\sim}{x}_{i_1}^{-\varepsilon_1''-1/2}(-1+\underset{\sim}{u}_1^{-1}) \ldots\ldots (-1+\underset{\sim}{u}_r^{-1}) .$$

Since, by hypothesis, a term of $x_{i_1 i_1}(C^{\delta})$ contains $\underset{\sim}{x}_{i_2}^{\varepsilon_2(r_2+1)}$ $(r_2 \geq 1)$ as a factor, it follows that in $\{u_1, \ldots, u_r\}$ or $\{u_1^{-1}, \ldots, u_r^{-1}\}$, there are at least $2x_{i_2}^{\varepsilon_2}$'s and in particular, $u_i = u_j = x_{i_2}^{\varepsilon_2}$ or $u_i^{-1} = u_j^{-1} = x_{i_2}^{\varepsilon_2}$ for some $i \neq j$. Thus, by 2.4.17, either

$$C^{\delta} = [x_{i_1}^{-\varepsilon_1'}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{\varepsilon_1''}, x_{i_2}^{\varepsilon_2}, u_3, \ldots, u_r]^{\delta}$$

or

$$C^{\delta} = [x_{i_1}^{-\varepsilon_1'}, x_{i_2}^{\varepsilon_2}; x_{i_1}^{\varepsilon_1''}, x_{i_2}^{-\varepsilon_2}, u_3, \ldots, u_r]^{\delta} .$$

Now, it follows that

either in $\{u_3, \ldots, u_r\}$ or in $\{u_3^{-1}, \ldots, u_r^{-1}\}$, there are

$(r_2-1)x_{i_2}^{\varepsilon_2}$'s, $(r_3+1)x_{i_3}^{\varepsilon_3}$'s, ..., $(r_{t_4}+1)x_{i_{t_4}}^{\varepsilon_{t_4}}$'s . In each case I get in turn

$$r-2 \geq r^*-4; \quad r+2 \geq r^*; \quad r+2 = r^* \quad (\text{since } r + 2 \leq r^*).$$

This now gives that $\varepsilon_1'' = -\varepsilon_1'$ and

either $\delta = -\delta_1$, so that

$$C^{\delta} = [x_{i_1}^{-\varepsilon_1'}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{-\varepsilon_1'}, x_{i_2}^{\varepsilon_2}, (r_2-1)x_{i_2}^{\varepsilon_2}, (r_3+1)x_{i_3}^{\varepsilon_3}, \ldots, (r_{t_4}+1)x_{i_{t_4}}^{\varepsilon_{t_4}}]^{-\delta_1}$$

$$= [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{-\varepsilon_1}, x_{i_2}^{\varepsilon_2}, (r_2-1)x_{i_2}^{\varepsilon_2}, (r_3+1)x_{i_3}^{\varepsilon_3}, \ldots, (r_{t_4}+1)x_{i_{t_4}}^{\varepsilon_{t_4}}]^{-\delta_1}$$

by 1.4.8,

or $\delta = \delta_1$, so that

$$C^{\delta} = [x_{i_1}^{-\varepsilon_1'}, x_{i_2}^{\varepsilon_2}; x_{i_1}^{-\varepsilon_1'}, x_{i_2}^{-\varepsilon_2}, (r_2-1)x_{i_2}^{-\varepsilon_2}, (r_3+1)x_{i_3}^{-\varepsilon_3}, \ldots, (r_{t_4}+1)x_{i_{t_4}}^{-\varepsilon_{t_4}}]^{\delta_1}$$

$$= [x_{i_1}^{-\varepsilon_1}, x_{i_2}^{-\varepsilon_2}; x_{i_1}^{-\varepsilon_1}, x_{i_2}^{\varepsilon_2}, (r_2-1)x_{i_2}^{\varepsilon_2}, (r_3+1)x_{i_3}^{\varepsilon_3}, \ldots, (r_{t_4}+1)x_{i_{t_4}}^{\varepsilon_{t_4}}]^{-\delta_1}$$

by 1.4.8, 1.2.1, 1.4.6 and 1.4.4.

In each case, I have $C_1^{\delta_1} C^{\delta} = e$ modulo $[F'',F]$, which contradicts the minimality of w_{14}. This completes the details of Step IV and hence also completes the proof of the main theorem. ○

REMARK. It is clear from the proof of the main theorem that if w is a power product of special commutators of Cat. II, III and IV only then $w \in [F'',F]$. Thus, it follows that for the free centre-extended-by-metabelian group of rank 3, the matrix representation is faithful.

CHAPTER III.

AN EXAMPLE

In this Chapter, I give an example of a centre-extended-by-metabelian group of class 6 which does not satisfy the law

2.1.16 $$[x^{-1},y^{-1};u,v][x^{-1},v^{-1};y,u][x^{-1},u^{-1};v,y].$$
$$.[v^{-1},y^{-1};x,u][y^{-1},u^{-1};x,v][u^{-1},v^{-1};x,y] .$$

3.1 CONSTRUCTION

Let $A = gp\{a_2,a_5,a_{10},a_{11},\ldots,a_{16},a_{18},\ldots,a_{34}\}$ be an elementary abelian 2-group of order 2^{26}. There are commuting automorphisms $\alpha_1,\alpha_3,\alpha_4,\alpha_6,\alpha_7,\alpha_8,\alpha_9$ and α_{17} of A each of order 2 given by,

α_1: $a_{25} \longrightarrow a_{25}a_{34}$, $a_{28} \longrightarrow a_{28}a_{34}$, $a_i \longrightarrow a_i$ for $i \neq 25, 28$;

α_3: $a_{23} \longrightarrow a_{23}a_{34}$, $a_i \longrightarrow a_i$ for $i \neq 23$;

α_4: $a_{22} \longrightarrow a_{22}a_{34}$, $a_{26} \longrightarrow a_{26}a_{34}$, $a_i \longrightarrow a_i$ for $i \neq 22, 26$;

α_6: $a_{21} \longrightarrow a_{21}a_{34}$, $a_i \longrightarrow a_i$ for $i \neq 21$;

α_7: $a_{18} \longrightarrow a_{18}a_{34}$, $a_i \longrightarrow a_i$ for $i \neq 18$;

α_8: $a_{16} \longrightarrow a_{16}a_{34}$, $a_i \longrightarrow a_i$ for $i \neq 16$;

α_9: $a_{15} \longrightarrow a_{15}a_{34}$, $a_{10} \longrightarrow a_{10}a_{34}$, $a_i \longrightarrow a_i$ for $i \neq 15, 10$;

α_{17}: $a_{14} \longrightarrow a_{14}a_{34}$, $a_i \longrightarrow a_i$ for $i \neq 14$.

Thus, the group generated by $\{\alpha_1,\alpha_3,\alpha_4,\alpha_6,\alpha_7,\alpha_8,\alpha_9,\alpha_{17}\}$ is an elementary abelian 2-group of order 2^8.

Let K_1 be the splitting extension of A by an elementary abelian 2-group B of order 2^8 generated by $a_1, a_3, a_4, a_6, a_7, a_8, a_9, a_{17}$, such that a_j induces α_j on A for $j = 1,3,4,6,7,8,9,17$. Then the group $K_1 = gp\{A,B\}$ is of order 2^{34} and has the following relations in addition to those of A and B:

$a_{25}^{a_1} = a_{25}a_{34}$, $a_{28}^{a_1} = a_{28}a_{34}$, $a_i^{a_1} = a_i$ for $i \neq 25, 28$;

$a_{23}^{a_3} = a_{23}a_{34}$, $a_i^{a_3} = a_i$ for $i \neq 23$;

$a_{22}^{a_4} = a_{22}a_{34}$, $a_{26}^{a_4} = a_{26}a_{34}$, $a_i^{a_4} = a_i$ for $i \neq 22, 26$;

$a_{21}^{a_6} = a_{21}a_{34}$, $a_i^{a_6} = a_i$ for $i \neq 21$;

$a_{18}^{a_7} = a_{18}a_{34}$, $a_i^{a_7} = a_i$ for $i \neq 18$;

$a_{16}^{a_8} = a_{16}a_{34}$, $a_i^{a_8} = a_i$ for $i \neq 16$;

$a_{15}^{a_9} = a_{15}a_{34}$, $a_{10}^{a_9} = a_{10}a_{34}$, $a_i^{a_9} = a_i$ for $i \neq 15, 10$;

$a_{14}^{a_{17}} = a_{14}a_{34}$, $a_i^{a_{17}} = a_i$ for $i \neq 14$.

I extend K_1 by adjoining an element a with the relations

$a^4 = 1$, $a_1^a = a_1a_7$, $a_2^a = a_2a_{11}$, $a_3^a = a_3a_{14}$, $a_4^a = a_4a_9a_{12}$,

$a_5^a = a_5a_{10}a_{15}$, $a_6^a = a_6a_{13}a_{16}$, $a_7^a = a_7$, $a_8^a = a_8a_{21}$,

$a_9^a = a_9$, $a_{10}^a = a_{10}a_{22}$, $a_{11}^a = a_{11}$, $a_{12}^a = a_{12}$, $a_{13}^a = a_{13}$,

$a_{14}^{a} = a_{14}$,, $a_{15}^{a} = a_{15}a_{26}$, $a_{16}^{a} = a_{16}a_{27}$, $a_{17}^{a} = a_{17}a_{23}$,

$a_{18}^{a} = a_{18}a_{25}$, $a_{19}^{a} = a_{19}a_{24}$, $a_{20}^{a} = a_{20}a_{25}a_{28}a_{34}$,

$a_{21}^{a} = a_{21}$, $a_{22}^{a} = a_{22}$, $a_{23}^{a} = a_{23}$, $a_{24}^{a} = a_{24}a_{31}$, $a_{25}^{a} = a_{25}$

$a_{26}^{a} = a_{26}$, $a_{27}^{a} = a_{27}$, $a_{28}^{a} = a_{28}a_{33}$, $a_{29}^{a} = a_{29}a_{32}a_{34}$,

$a_{30}^{a} = a_{30}a_{32}$, $a_{31}^{a} = a_{31}$, $a_{32}^{a} = a_{32}a_{34}$, $a_{33}^{a} = a_{33}$,

$a_{34}^{a} = a_{34}$.

It is easily checked that the transformtion by a induces an automorphism of order 4 on K_1, so that $K_2 = gp\{K_1, a\}$ is a splitting extension of K_1 by the cyclic group $\{a\}$.

Next, I extend K_2 by adjoining an element b with the relations

$b^4 = 1$, $a^b = aa_1$, $a_1^b = a_1a_8$, $a_2^b = a_2a_{12}$, $a_3^b = a_3a_{15}$,

$a_4^b = a_4a_{17}$, $a_5^b = a_5a_{19}$, $a_6^b = a_6a_{18}a_{20}$, $a_7^b = a_7a_{21}$,

$a_8^b = a_8$, $a_9^b = a_9a_{23}$, $a_{10}^b = a_{10}a_{24}$, $a_{11}^b = a_{11}$, $a_{12}^b = a_{12}$,

$a_{13}^b = a_{13}$, $a_{14}^b = a_{14}a_{26}$, $a_{15}^b = a_{15}$, $a_{16}^b = a_{16}a_{28}$,

$a_{17}^b = a_{17}$, $a_{18}^b = a_{18}a_{29}$, $a_{19}^b = a_{19}$, $a_{20}^b = a_{20}a_{30}$, $a_{21}^b = a_{21}$,

$a_{22}^b = a_{22}a_{31}$, $a_{23}^b = a_{23}$, $a_{24}^b = a_{24}$, $a_{25}^b = a_{25}a_{32}$,

$a_{26}^b = a_{26}$, $a_{27}^b = a_{27}a_{33}a_{34}$, $a_{28}^b = a_{28}$, $a_{29}^b = a_{29}$,

$a_{30}^b = a_{30}$, $a_{31}^b = a_{31}$, $a_{32}^b = a_{32}$, $a_{33}^b = a_{33}a_{34}$,

$a_{34}^b = a_{34}$;

it is easily checked that the transformation by b induces an automorphism of order 4 on K_2, so that $K_3 = gp\{K_2, b\}$ is a splitting extension of K_2 by the cyclic group $\{b\}$.

Further, I extend K_3 by an element c with the following relations;

$c^2 = 1$, $a^c = aa_2$, $b^c = ba_4$, $a_1^c = a_1a_9$, $a_2^c = a_2$, $a_3^c = a_3a_{16}$,

$a_4^c = a_4$, $a_5^c = a_5a_{20}$, $a_6^c = a_6$, $a_7^c = a_7$, $a_8^c = a_8a_{23}$, $a_9^c = a_9$,

$a_{10}^c = a_{10}a_{25}a_{34}$, $a_{11}^c = a_{11}$, $a_{12}^c = a_{12}$, $a_{13}^c = a_{13}$, $a_{14}^c = a_{14}a_{27}$,

$a_{15}^c = a_{15}a_{28}$, $a_{16}^c = a_{16}$, $a_{17}^c = a_{17}$, $a_{18}^c = a_{18}$, $a_{19}^c = a_{19}a_{30}$,

$a_{20}^c = a_{20}$, $a_{21}^c = a_{21}$, $a_{22}^c = a_{22}$, $a_{23}^c = a_{23}$, $a_{24}^c = a_{24}a_{32}$,

$a_{25}^c = a_{25}$, $a_{26}^c = a_{26}a_{33}$, $a_{27}^c = a_{27}$, $a_{28}^c = a_{28}$, $a_{29}^c = a_{29}$,

$a_{30}^c = a_{30}$, $a_{31}^c = a_{31}a_{34}$, $a_{32}^c = a_{32}$, $a_{33}^c = a_{33}$, $a_{34}^c = a_{34}$;

it is easily checked that the transformation by c induces an automorphism of order 2 on K_3. Thus, $K_4 = gp\{K_3, c\}$ is a splitting extension of K_3 by the cyclic group $\{c\}$.

Finally, I extend K_4 by an element d with the relations

$d^2 = 1$, $a^d = aa_3$, $b^d = ba_5$, $c^d = ca_6$, $a_1^d = a_1a_{10}$,

$a_2^d = a_2a_{13}$, $a_3^d = a_3$, $a_4^d = a_4a_{18}$, $a_5^d = a_5$, $a_6^d = a_6$,

$a_7^d = a_7a_{22}$, $a_8^d = a_8a_{24}$, $a_9^d = a_9a_{25}$, $a_{10}^d = a_{10}$,

$a_{11}^d = a_{11}$, $a_{12}^d = a_{12}$, $a_{13}^d = a_{13}$, $a_{14}^d = a_{14}$, $a_{15}^d = a_{15}$,

$a_{16}^d = a_{16}$, $a_{17}^d = a_{17}a_{29}$, $a_{18}^d = a_{18}$, $a_{19}^d = a_{19}$, $a_{20}^d = a_{20}$,

$a_{21}^d = a_{21}a_{31}$, $a_{22}^d = a_{22}$, $a_{23}^d = a_{23}a_{32}$,

$a_j^d = a_j$ for $j \in \{24,\ldots,34\}$;

the transformation of K_4 by d induces an automorphism of order 2 on K_4. Thus, the group $H = gp\{K_4,d\}$ is a splitting extension of K_4 by the cyclic group $\{d\}$.

From the above the relations, it is easily seen that H is nilpotent-of-class-6 and is generated by a,b,c,d.

With the notation: $a = 1$, $b = 2$, $c = 3$, $d = 4$, $[a,b] = 12$, $[[a,b],c] = 123$, $[a,b;c,d] = [12;34]$ etc; the following commutator relations hold in H,

$21 = a_1$, $31 = a_2$, $41 = a_3$, $32 = a_4$, $42 = a_5$, $43 = a_6$,

$211 = a_7$, $212 = a_8$, $213 = a_9$, $214 = a_{10}$, $311 = a_{11}$,

$312 = a_{12}$, $314 = a_{13}$, $411 = a_{14}$, $412 = a_{15}$, $413 = a_{16}$,

$322 = a_{17}$, $324 = a_{18}$, $422 = a_{19}$, $423 = a_{20}$,

$2112 = a_{21}$, $2114 = a_{22}$, $2123 = a_{23}$, $2124 = a_{24}$,

$2134 = a_{25}$, $4112 = a_{26}$, $4113 = a_{27}$, $4123 = a_{28}$,

$3224 = a_{29}$, $4223 = a_{30}$, $21124 = a_{31}$, $21234 = a_{32}$, $41123 = a_{33}$,

$$\begin{aligned} a_{34} &= [2134;21] = [4123;21] = [2123;41] \\ &= [4112;32][= [2114;32] = [2112;43] \\ &= [213;214] = [412;213] = [212;413] \\ &= [411;322] = [324;211]. \end{aligned}$$

With the usual ordering $1 < 2 < 3 < 4$, all commutators in the above relations are basic (in the sense of Marshall Hall [9]).

H is centre-extended-by-metabelian.

In H, all basic commutators of weight 4 and 5 which belong to H'' are trivial, so that H'' is generated by basic commutators of weight 6 only and these are central in H.

H does not satisfy 2.1.16.

In the expression 2.1.16, take $x = a$, $y = b$, $u = c$ and $v = d$.

Since $[a^{-1},b^{-1}] = [a,b]^{a^{-1}b^{-1}} = [a,b][a,b,a^{-1}b^{-1}]$ by 1.2.6, 1.2.2

$$= [a,b][a,b,b^{-1}][a,b,a^{-1}][a,b,a^{-1},b^{-1}]$$

by 1.2.5

and

$$[a,b,b^{-1}] = [a,b,b]^{-b^{-1}} = [a,b,b],$$

$$[a,b,a^{-1}] = [a,b,a],$$

$$[a,b,a^{-1},b^{-1}] = [a,b,a,b],$$

I have

$$\begin{aligned}[a^{-1},b^{-1};c,d] &= [a,b,a,b;c,d] \\ &= [b,a,a,b;d,c] \\ &= a_{34} \neq 1.\end{aligned}$$

Since

$$\begin{aligned}[a^{-1},d^{-1}] = [a^{-1},d] &= [a,d]^{-a^{-1}} \qquad \text{by 1.2.6, 1.2.1} \\ &= [a,d][a,d,a],\end{aligned}$$

I have

$$[a^{-1},d^{-1};b,c] = 1 .$$

Similarly I obtain

$$[a^{-1},c^{-1};d,b] = 1,\ [d^{-1},b^{-1};a,c] = 1,$$

$$[b^{-1},c^{-1};a,d] = 1 \text{ and } [c^{-1},d^{-1};a,b] = 1.$$

Thus, the elements a,b,c,d of H do not satisfy 2.1.16.

This example now tells us that the variety $\underline{\underline{V}}$ (see page 21) is a proper sub-variety of the variety of centre-extended-by-metabelian groups.

<u>REMARK.</u> Incidentally, the above example shows that if U is the free centre-extended-by-metabelian group then the factor group

$\Gamma_6(U)/\Gamma_7(U)$ [5] has an element of order 2. This is in accordance with a verbal communication by Professor Hanna Neumann that a student of K.W. Gruenberg has observed that $\Gamma_{2n}(U)/\Gamma_{2n+1}(U)$ has an element of order 2 for all $n \geq 3$. In fact, an easy 2-generator example can be constructed to confirm this observation. However, I shall not go into details about this.

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5) $\Gamma_m(U)$ is the m-th term of the lower central series of U.

CHAPTER IV.

SOME PROPERTIES OF CENTRE-EXTENDED-BY-METABELIAN GROUPS

In this Chapter, I observe some properties of finitely generated centre-extended-by-metabelian groups while imposing some conditions on its 2-generator subgroups.

It is known that every finitely generated soluble Engel group [6)] is nilpotent (Theorem 1, K.W. Gruenberg [6]) and that every finitely generated nilpotent group satisfies the maximal condition [7)] for subgroups and is residually finite [8)]. In fact, finitely generated nilpotent groups are polycyclic [9)] (Lemma 2, P. Hall [10]) which are known to satisfy both the maximal condition for subgroups (p.426, P. Hall [10]) and the residual

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6) A group H is said to be an Engel group if to every pair of elements a,b in H there exists an integer k (depending upon a and b) such that $[a,kb] = 1$.

7) A group is said to satisfy the maximal condition for subgroups if every properly ascending chain of its subgroups is finite.

8) A group H is called residually finite if to each element $h \neq 1$ in H there exists a normal subgroup N_h such that H/N_h is finite and $h \notin N_h$.

9) A group H is said to be polycyclic if it can be obtained from the unit subgroup 1 by a finite number of cyclic extensions; or equivalently, if H is a soluble group such that the derived factors H/H', H'/H'', ... are all finitely generated.

finiteness condition (p.37, K.W. Gruenberg [7]).

Thus, it follows that a finitely generated centre-extended-by-metabelian Engel group (and hence also a finitely generated centre-extended-by-metabelian group with every 2-generator subgroup nilpotent) satisfies the maximal condition for subgroups and is residually finite.

Next, I impose the condition that every 2-generator subgroup is metabelian. Here I am able to prove the following:

<u>THEOREM 4.1</u> A finitely generated centre-extended-by-metabelian group, all of whose 2-generator subgroups are metabelian, satisfies the maximal condition for normal subgroups.

Another related condition is that the group satisfies the identity $[x,y,y^2] = 1$.[10)] Here I prove

<u>THEOREM 4.2</u> A finitely generated centre-extended-by-metabelian group with the identity $[x,y,y^2] = 1$ is polycyclic.

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10) All dihedral groups satisfy $[x,y,y^2] = 1$, so there are non-nilpotent groups satisfying this identity.

PROOF OF THEOREM 4.1

Let H be a group satisfying the hypothesis of theorem 4.1. Then H satisfies the identities

$$[x,y;u,v;z] = 1 \quad \text{and} \quad [x,y;x,y^{-1}] = 1.$$

I first of all prove that these two identities imply the identity

4.1.1 $$[x,y;u,ny,k_1v_1,\ldots,k_sv_s]$$

$$= [x,y;u,2y,k_1v_1,\ldots,k_sv_s]^{(-2)^{n-2}} \qquad (k_i \geq 0,\ n \geq 2)$$

for all $x,y,u,v_1,\ldots,v_s$ in H.

Proof of 4.1.1.

$$\begin{aligned}
\text{Since,}\quad 1 &= [x,y;x,y^{-1}] \\
&= [xu,y;xu,y^{-1}] \\
&= [[x,y]^u[u,y];[x,y^{-1}]^u[u,y^{-1}]] && \text{by 1.2.4} \\
&= [[x,y]^u;[u,y^{-1}]][[u,y];[x,y^{-1}]^u] && \text{by 1.2.4, 1.2.5} \\
&= [[x,y];[u,y^{-1}]^{u^{-1}}]^u[[u,y]^{u^{-1}};[x,y^{-1}]]^u && \text{by 1.2.3} \\
&= [[x,y];[u^{-1},y^{-1}]^{-1}][[u^{-1},y]^{-1};[x,y^{-1}]] && \text{by 1.2.6, 1.2.1} \\
&= [[x,y];[y^{-1},u^{-1}]][[y,u^{-1}];[x,y^{-1}]] && \text{by 1.2.1} \\
&= [[x,y];[y^{-1},u]][[y,u];[x,y^{-1}]] \ ,
\end{aligned}$$

I have

$$[x,y;y^{-1},u] = [x,y^{-1};y,u] \qquad \text{by 1.2.1}$$

$$= [[x,y]^{-y^{-1}};[y,u]] \qquad \text{by 1.2.6, 1.2.1}$$

$$= [[x,y]^{-1};[y,u]^{y}]^{y^{-1}} \qquad \text{by 1.2.3}$$

$$= [[x,y];[y,u]^{y}]^{-[x,y]^{-1}} \qquad \text{by 1.2.6, 1.2.1}$$

$$= [x,y;[u,y][u,y,y]] \qquad \text{by 1.4.3, 1.2.2}$$

on the other hand,

$$[x,y;y^{-1},u] = [x,y;[u,y]^{y^{-1}}] \qquad \text{by 1.2.6}$$

$$= [x,y;[u,y][u,y,y^{-1}]] \qquad \text{by 1.2.2.}$$

Thus

$$[x,y;u,y,y^{-1}] = [x,y;u,y,y],$$

and in turn

$$[x,y;u,y,y,y^{-1}] = [x,y;u,y,y]^{-2} \qquad \text{by 1.4.3}$$

(since $[u,y,y^{-1}] = [u,y,y]^{-y^{-1}} = [u,y,y,y^{-1}]^{-1}[u,y,y]^{-1}$
by 1.2.6, 1.2.1, 1.2.2);

$$[x,y;u,y,y,y^{-1}] = [x,y;u,y,y,y]$$

(replacing u by $[u,y]$ in $[x,y;u,y,y^{-1}] = [x,y;u,y,y]$);

$$[x,y;u,3y] = [x,y;u,2y]^{-2};$$

and more generally

$$[x,y;u,ny] = [x,y;u,2y]^{(-2)^{n-2}}.$$

Replacing u by uv_1 in above gives

$$[x,y;u,ny,v_1] = [x,y;u,2y,v_1]^{(-2)^{n-2}} \qquad \text{by 1.2.4, 1.4.1}$$

and similar repeated arguments finally give

$$[x,y;u,ny,k_1v_1,\ldots,k_sv_s] = [x,y;u,2y,k_1v_1,\ldots,k_sv_s]^{(-2)^{n-2}},$$

as was required.

Next I prove

4.1.2 If $n \geq 3$, then

$$[x,u;v,z,ny,k_1v_1,\ldots,k_rv_r]$$
$$= [x,u;v,z,2y,y^{-1},k_1v_1,\ldots,k_rv_r]^{(-2)^{n-3}} \quad (k_i \geq 0).$$

Proof of 4.1.2.

Since, $[x,u;v,z,(n-1)y,yy^{-1},k_1v_1,\ldots,k_rv_r]$

$$= [x,u;v,z,(n-1)y,y^{-1},k_1v_1,\ldots,k_rv_r]$$

$$[x,u;v,z,(n-1)y,y,k_1v_1,\ldots,k_rv_r]$$

$$[x,u;v,z,(n-1)y,y,y^{-1},k_1v_1,\ldots,k_rv_r] \qquad \text{by 1.2.5, 1.4.4,}$$

it follows that

$$[x,u;v,z,ny,k_1v_1,\ldots,k_rv_r]$$

$$= [x,u;v,z,(n-1)y,y^{-1},k_1v_1,\ldots,k_rv_r]^{-1}$$

$$[x,u;v,z,ny,y^{-1},k_1v_1,\ldots,k_rv_r]^{-1}$$

$$= [x,u,y;v,z,(n-1)y,k_1v_1,\ldots,k_rv_r]^{-1}$$

$$[x,u,y;v,z,ny,k_1v_1,\ldots,k_rv_r]^{-1} \qquad \text{by 1.4.4, 1.4.6}$$

$$= [x,u,y;v,z,2y,k_1v_1,\ldots,k_rv_r]^{-(-2)^{n-3}}$$

$$[x,u,y;v,z,2y,k_1v_1,\ldots,k_rv_r]^{-(-2)^{n-2}} \qquad \text{by 4.1.1}$$

$$= [x,u;v,z,2y,y^{-1},k_1v_1,\ldots,k_rv_r]^{(-2)^{n-3}} \qquad \text{by 1.4.6, 1.4.4.}$$

Now, I am in a position to prove the theorem 4.1. First of all, I prove that H'' is finitely generated. Let H be generated by a finite set $X = \{x_1,\ldots,x_n\}$ $(n \geq 3)$, then the set S, of all commutators in H'' of weight at most $6n$ with entries from the set XUX^{-1}, is finite; and so it is sufficient to prove that every commutator in H'' of weight greater than $6n$ with entries from the set XUX^{-1}, can be written as a power product of commutators in S.

Let C be a commutator of weight greater than $6n$, then for some i, either x_i or x_i^{-1} occurs at least 4 times in C. Suppose x_i occurs at least 4 times in C. By 1.4.6, each commutator in H'' can be written in the form

$$[y_1,y_2;y_3,y_4,\ldots,y_r] \quad (r \geq 4) ,$$

where $y_i \in XUX^{-1}$

If $x_i \in \{y_1, y_2\}$, then at least $3x_i$'s are among $y_3, \ldots, y_r$. By 4.1.1, $[y_1, y_2; y_3, y_4, \ldots, y_r]$ can be written as a power product of commutators of smaller weight each containing at most $3x_i$'s.

If $x_i \notin \{y_1, y_2\}$, then at least $4x_i$'s are among $y_3, \ldots, y_r$. By 4.1.6 and/or 4.1.3, $[y_1, y_2; y_3, y_4, \ldots, y_r]$ can be written as a power product of commutators of smaller weight each containing at most $3x_i$'s.

Using the argument for each x_i and x_i^{-1} which occurs more than four times in $[y_1, y_2, ; y_3, y_4, \ldots, y_r]$, I conclude that each such commutator can be written as a power product of commutators of weight at most $6n$ and hence H'' is finitely generated. Since H'' is a finitely generated abelian normal subgroup of H, it satisfies the maximal condition for subgroups and hence for normal subgroups.

Further, since H/H'' is finitely generated metabelian group, by P. Hall ([10], p,430), it satisfies the maximal condition for normal subgroups. Since, H is an extension of H'' by H/H'', it follows by P. Hall ([10], p.426), that H itself satisfies the maximal condition for normal subgroups. This completes the proof of the Theorem 4.1.

PROOF OF THEOREM 4.2

Let H be a group satisfying the hypothesis of theorem 4.2. I first of all prove

4.2.1 $[x,y,y^2] = 1$ for all x,y in H implies

$$[x,y;x,y^{-1}] = 1 \text{ for all } x,y \text{ in } H.$$

Proof. Since, $1 = [x,y,y^2]$,

I have

$$[x,y] = [x,y]^{y^2} .$$

Replacing y by xy, I get

$$[x,y] = [x,y]^{xyxy} = [x,y]^{x^2(x^{-1}yxy^{-1})y^2} ,$$

and transforming by y^{-2} on both sides gives

$$[x,y]^{y^{-2}} = [x,y]^{x^2[x,y^{-1}]}$$

which implies that

$$[x,y] = [x,y]^{[x,y^{-1}]}$$

(because, $[x,y,y^2] = 1$ implies $[x,y,y^{-2}] = 1$, and also $[x,y,x^2] = 1$).

Thus, by 1.2.2, I have $[x,y;x,y^{-1}] = 1$, as required.

By a well-known result of G. Higman [13], the identity $[x,y;x,y^{-1}] = 1$ in a group is equivalent to the property that every

2-generator subgroup is metabelian. Thus, the hypothesis of Theorem 4.1 applies and, in H, the subgroup H'' is finitely generated. It remains to show that H'/H'' is finitely generated.

Next I prove in H,

4.2.2 For $n \geq 2$, $k_i \geq 0$,

$$[u,ny,k_1z_1,\ldots,k_rz_r]$$

$$= [u,2y,k_1z_1,\ldots,k_rz_r]^{(-2)^{n-2}} \quad \text{modulo } H''$$

for all $u,y,z_1,\ldots,z_r$ in H.

<u>Proof</u>. Since, $1 = [u,y,y^2] = [u,y,y]^2[u,y,y,y]$ by 1.2.5, I have $[u,3y] = [u,2y]^{-2}$, which on replacing u by uz_1 gives

$[uz_1,3y] = [uz_1,2y]^{-2}$, and on expanding gives

$$[u,3y][u,3y,z_1][z_1,3y] = [u,2y]^{-2}[[u,2y]^{-2},z_1][z_1,2y]^{-2} \quad \text{modulo } H''$$

by 1.2.4,

so that

$$[u,3y,z_1] = [u,2y,z_1]^{-2} \quad \text{modulo } H''.$$

Similar repeated arguments give

$$[u,3y,k_1z_1,\ldots,k_rz_r] = [u,2y,k_1z_1,\ldots,k_rz_r]^{-2} \quad \text{modulo } H'',$$

and more generally, I get

$$[u,ny,k_1z_1,\ldots,k_rz_r] = [u,2y,k_1z_1,\ldots,k_rz_r]^{(-2)^{n-2}} \quad \text{modulo } H''.$$

4.2.3 If $n \geq 2$, $k_i \geq 0$, then in H

$$[u,ny^{-1},k_1z_1,\ldots,k_rz_r]$$

$$= [u,ny,k_1z_1,\ldots,k_rz_r] \text{ modulo } H''.$$

Proof. Since, $[u,y,y^2] = 1$, I have

$$[u,y,y] = [u,y,y^{-1}] = [u,y]^{-1}[u,y^{-1}]^{-1} \qquad \text{by 1.4.7}$$

$$= [u,y^{-1}]^{-1}[u,y]^{-1} = [u,y^{-1},y]$$

$$= [u,y^{-1},y^{-1}] \quad \text{(by hypothesis)},$$

which on replacing u by uz_1 and expanding modulo H'' gives

$$[u,y,y,z_1] = [u,y^{-1},y^{-1},z_1] \qquad \text{by 1.2.4.}$$

Similarly,

$$[u,y,y,k_1z_1,\ldots,k_rz_r] = [u,y^{-1},y^{-1},k_1z_1,\ldots,k_rz_r] \quad \text{modulo } H''.$$

And more generally, I get

$$[u,ny,k_1z_1,\ldots,k_rz_r] = [u,ny^{-1},k_1z_1,\ldots,k_rz_r] \quad \text{modulo } H''.$$

Now, I am in a position to complete the proof of theorem 4.2.

Let H be generated by $X = \{x_1, \ldots, x_n\}$ $(n \geq 2)$, then the set $\overline{S}$ of all commutators in $H' \setminus H''$ of weight at most $2n$ with entries from XUX^{-1} is finite; therefore, it suffices to prove that every commutator in $H' \setminus H''$ of weight greater than $2n$ with entries from XUX^{-1} can be written as a power product of commutators in $\overline{S}$.

Let C be a commutator of weight greater than $2n$, then for some i, either x_i occurs at least 3 times or x_i^{-1} occurs at least 3 times or x_i occurs at least twice and x_i^{-1} occurs at least once or x_i^{-1} occurs at least twice and x_i occurs at least once.

In each case, by using 4.2.2 and 4.2.3, C can be written as a power product of commutators of smaller weight each containing at most $2x_i$'s.

Using the argument for all such i's, C can be written as a power product of commutators of weight at most $2n$ and hence H'/H'' is finitely generated, as was required.

$\bigcirc$

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